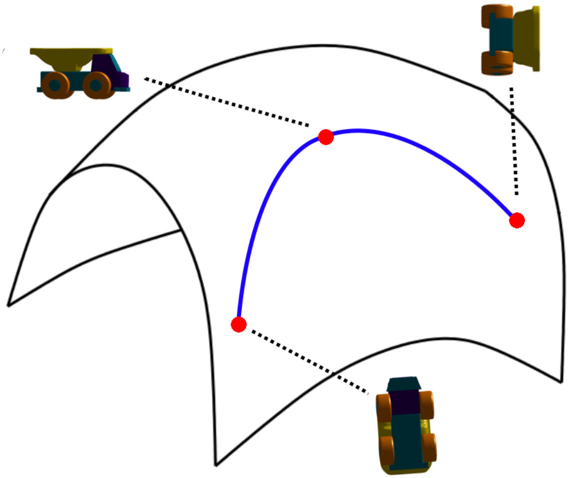
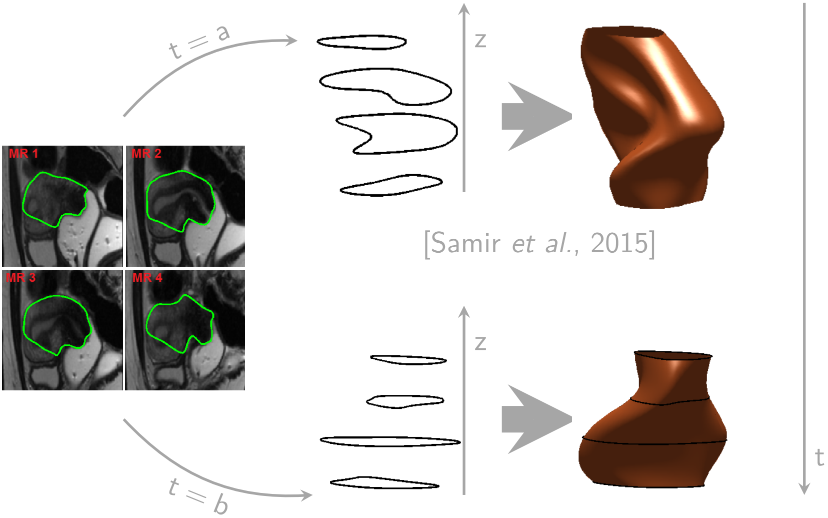
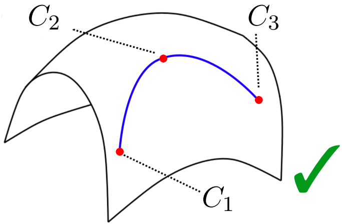
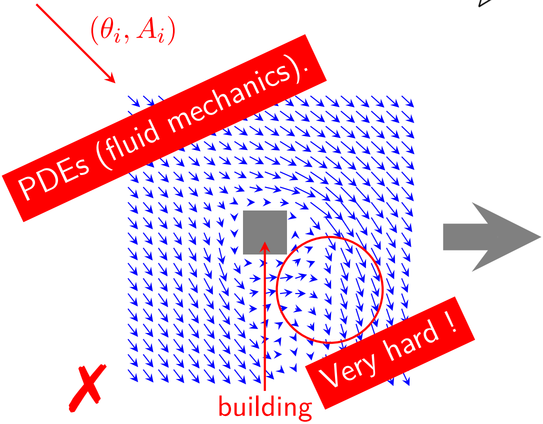


A medical application

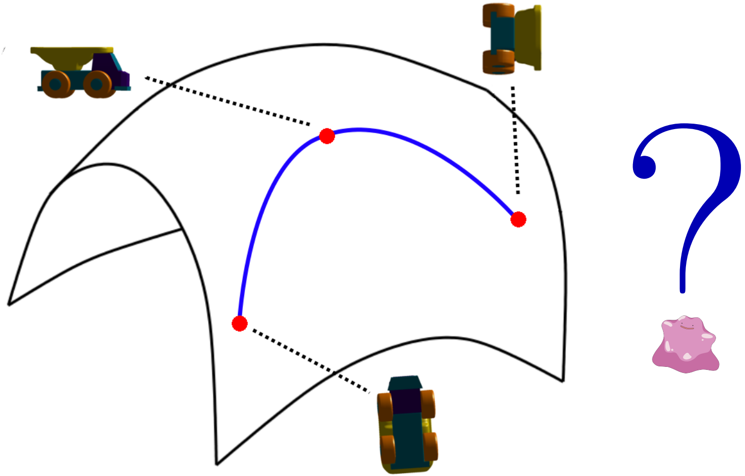


The wind field estimation





$C_i \in \mathcal{S}_+(r, n)$



How to interpolate or fit points on $\mathcal{M}$...

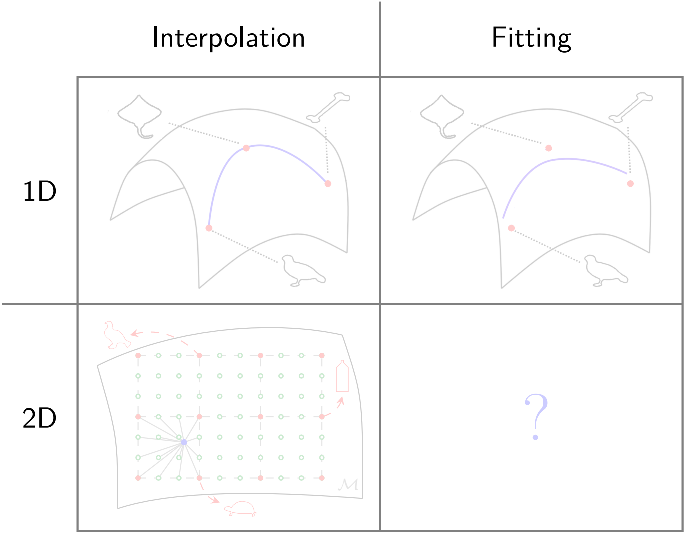
... in 1D and 2D ?

Interpolation and fitting on manifolds with differentiable piecewise-Bézier functions

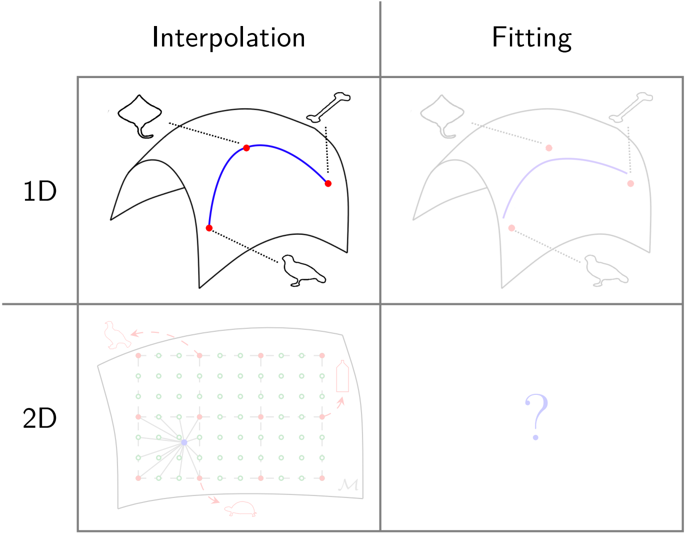
Pierre-Yves Gousenbourger
pierre-yves.gousenbourger@uclouvain.be

13 décembre 2018

The path...

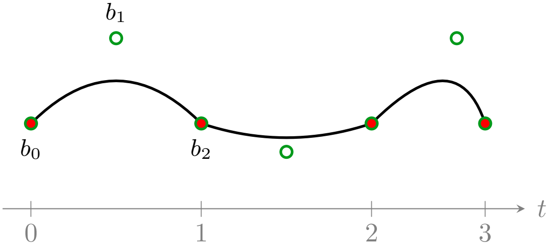


The path...



1D : Interpolative Bézier curves

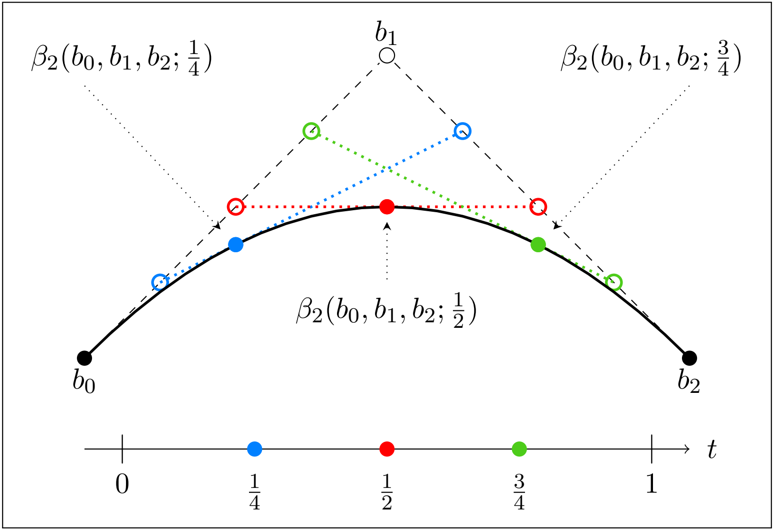
Each segment between two consecutive points is
a **Bézier curve** of degree K.



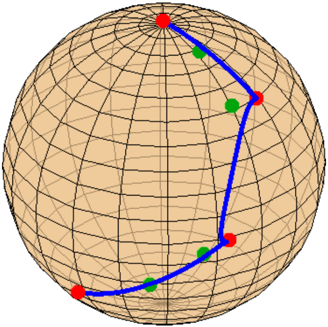
$$\beta_K(t, \mathbf{b}) = \sum_{i=0}^{K} b_i B_{iK}(t)$$

[G. *et al.* 2014, Arnould *et al.* 2015]

Reconstruction : the De Casteljau algorithm

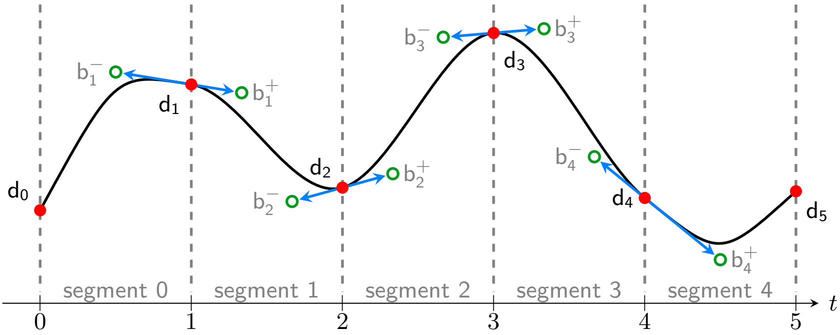


Example on the sphere



It’s ugly. Make it **smooth** !

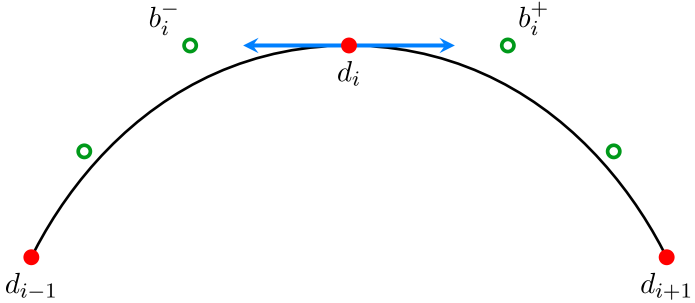
Smooth interpolation with Bézier (in $\mathbb{R}^n$)



Each segment is a Bézier curve smoothly connected!

Unknowns : b_i^-, b_i^+.

Differentiability



$$b_i^+ = 2d_i - b_i^-$$

Optimal $\mathcal{C}^1$-piecewise Bézier interpolation (in $\mathbb{R}^n$)

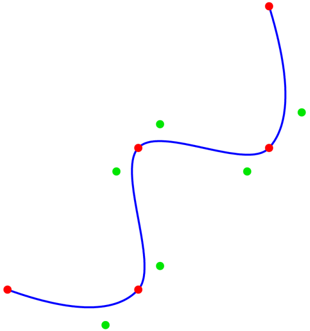
Minimization of the mean squared acceleration of the path

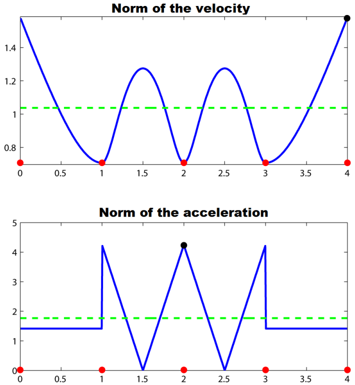
$$\min_{b_i^-} \int_0^1 \|\ddot{\beta}_2^0(b_1^-;t)\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(b_i^-;t)\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n(b_{n-1}^-;t)\|^2 \mathrm{d}t$$

$$\underbrace{\min_{b_i^-} \int_0^1 \|\ddot{\beta}_2^0(b_1^-;t)\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(b_i^-;t)\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n(b_{n-1}^-;t)\|^2 \mathrm{d}t}_{\text{Second order polynomial } P(b_i^-)}$$

$$\underbrace{\min_{b_i^-} \int_0^1 \|\ddot{\beta}_2^0(b_1^-;t)\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i(b_i^-;t)\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n(b_{n-1}^-;t)\|^2 \mathrm{d}t}_{\text{Second order polynomial } P(b_i^-)}$$

A result on $\mathbb{R}^2$





Optimal $\mathcal{C}^1$-piecewise Bézier interpolation (on $\mathcal{M}$)

- The control points are given by :

$$b_i^- = \sum_{j=0}^{n} q_{i,j} d_j$$

- These points are invariant under translation, *i.e.* :

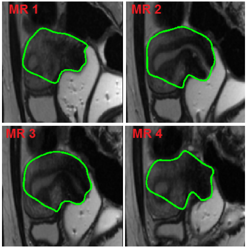
$$b_i^- - d^{ref} = \sum_{j=0}^{n} q_{i,j} (d_j - d^{ref})$$

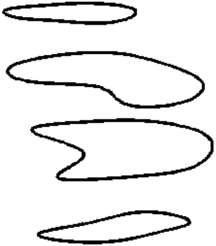
- On manifolds : projection to the **tangent space** of d^{ref} with the **Log**, as $a - b \Leftrightarrow \mathrm{Log}_b(a)$

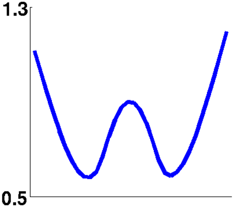
$$v_i = \mathrm{Log}_{d^{ref}}(b_i^-) = \sum_{j=0}^{n} q_{i,j} \mathrm{Log}_{d^{ref}}(d_j)$$

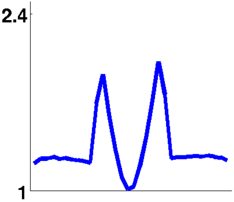
- Back to the manifold with the **Exp** : $b_i^- = \mathrm{Exp}_{d^{ref}}(v_i)$.

Application to MRI – the manifold of closed shapes









Interpolation with Bézier : pros and cons

✓ Optimality conditions are a closed form linear system.

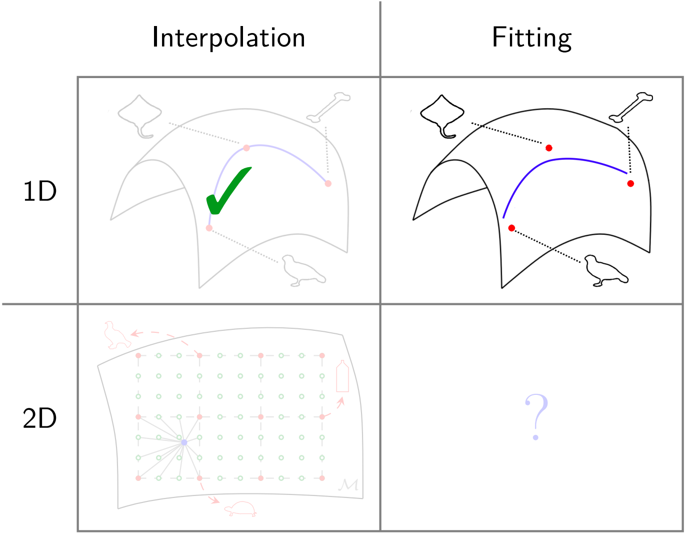
✓ Method only needs exp and log maps.

✓ The curve is $\mathcal{C}^1$.

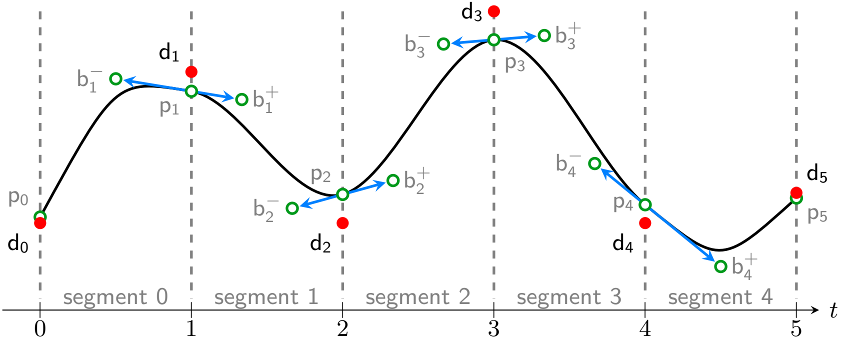
✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

[G. *et al.*, 2014] [Arnould *et al.*, 2015] [Pyta *et al.*, 2016]

The path...



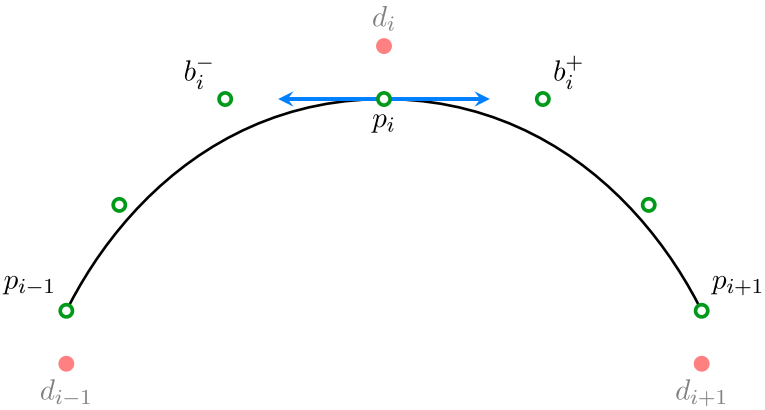
Smooth fitting with Bézier (in $\mathbb{R}^n$)



Now data points are approached but not interpolated !

Unknowns : b_i^-, b_i^+, p_i.

Differentiability



$$p_i = \frac{b_i^- + b_i^+}{2}$$

Optimal $\mathcal{C}^1$-piecewise Bézier fitting (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the path

$$\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2$$

$$\underbrace{\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2}_{\text{Second order polynomial } P(p_0, b_i^-, b_i^+, p_n, \lambda)}$$

$$\underbrace{\min_{p_0, b_i^-, b_i^+, p_n} \int_0^1 \|\ddot{\beta}_2^0\|^2 \mathrm{d}t + \sum_{i=1}^{n-1} \int_0^1 \|\ddot{\beta}_3^i\|^2 \mathrm{d}t + \int_0^1 \|\ddot{\beta}_2^n\|^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \|d_i - p_i\|_2^2}_{\text{Second order polynomial } P(p_0, b_i^-, b_i^+, p_n, \lambda)}$$

$$\nabla P(p_0, b_i^-, b_i^+, p_n)\,!$$

Optimal $\mathcal{C}^1$-piecewise Bézier fitting (on $\mathcal{M}$)

- The control points are given by :

$$x_i = \sum_{j=0}^{n} q_{i,j}(\lambda) d_j$$

- These points are invariant under translation, *i.e.* :

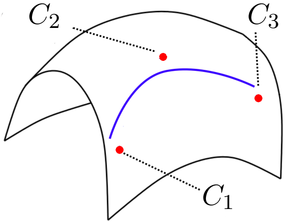
$$x_i - d^{ref} = \sum_{j=0}^{n} q_{i,j}(\lambda)(d_j - d^{ref})$$

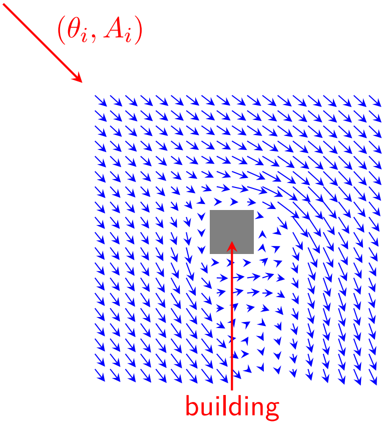
- On manifolds : projection to the **tangent space** of d^{ref} with the **Log**, as $a - b \Leftrightarrow \mathrm{Log}_b(a)$

$$v_i = \mathrm{Log}_{d^{ref}}(x_i) = \sum_{j=0}^{n} q_{i,j}(\lambda)\mathrm{Log}_{d^{ref}}(d_j)$$

- Back to the manifold with the **Exp** : $x_i = \mathrm{Exp}_{d^{ref}}(v_i)$, where $d^{ref} = d_i$ if x_i is b_i^-, p_i, b_i^+.

Application : Wind field estimation

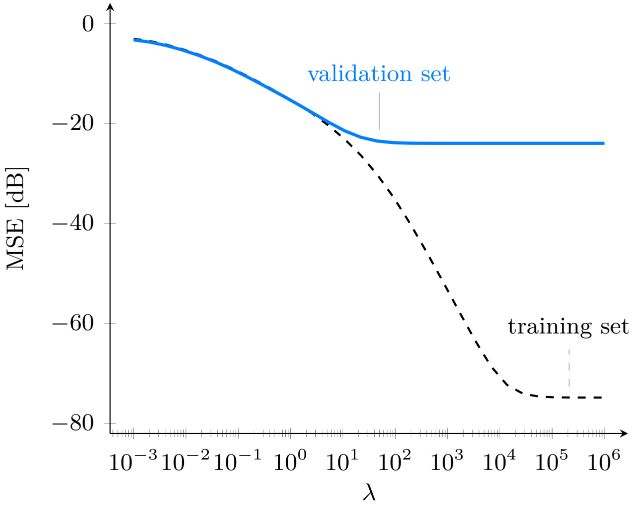




$C_i \in \mathcal{S}_+(r, n)$

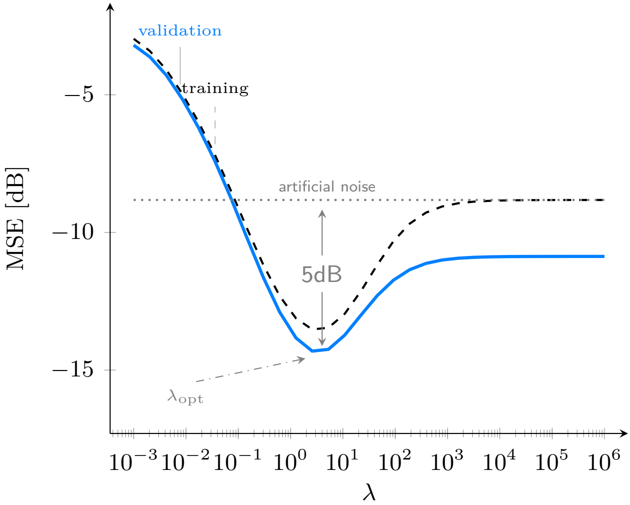
Application : Wind field estimation on $S_+(r, p)$.

No noise on data.



Application : Wind field estimation on $S_+(r,n)$.

With artificial noise (8dB) on data.



Fitting with Bézier : pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps.

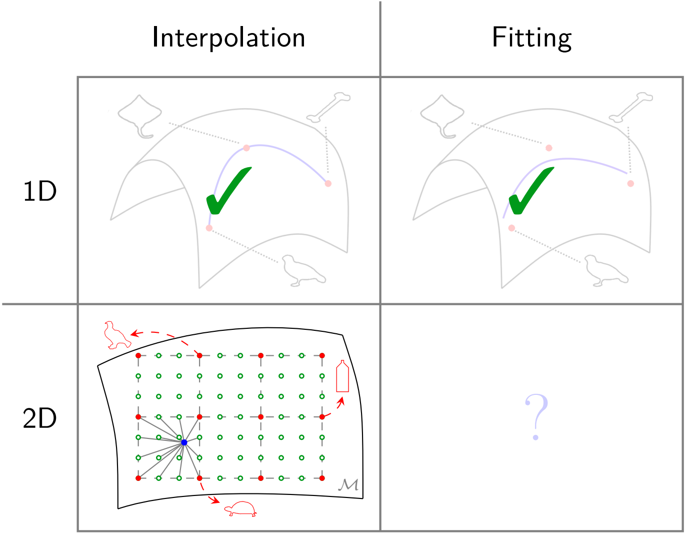
✓ The curve is $\mathcal{C}^1$.

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

✓ We can do denoising.

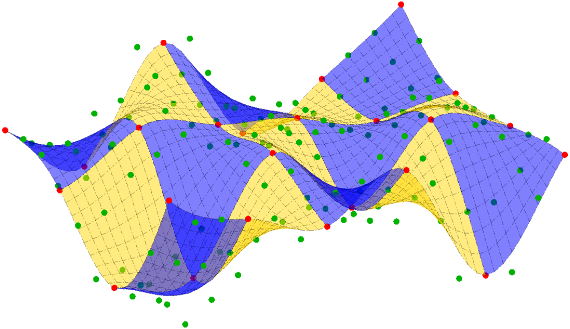
Paper submitted at the ESANN conference, 2017. Joint work with MIT.

The path...



2D : Interpolative Bézier surface

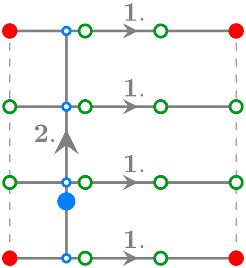
Each patch between four neighbour points is
a **Bézier surface** of degree K.



$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_{jK}(t_2)$$

Bézier surface on one patch

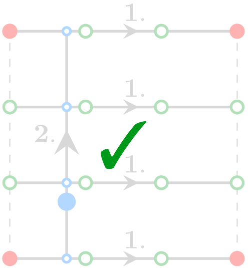
$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_{jK}(t_2) = \sum_{i=0}^{K} \tilde{b}_j B_{jK}(t_2)$$



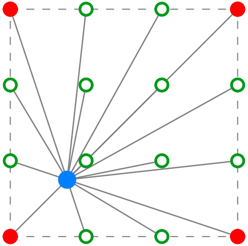
Two-curves

Bézier surface on one patch

$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K}\sum_{j=0}^{K} b_{ij} \overbrace{B_{iK}(t_1)B_{jK}(t_2)}^{w_{ij}} = \text{av}[\mathbf{b}, w_{ij}]$$



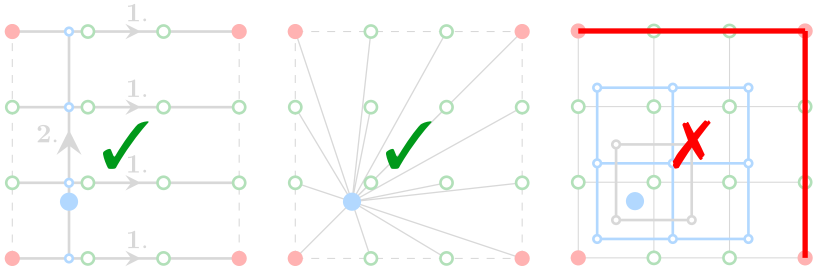
Two-curves



Karcher

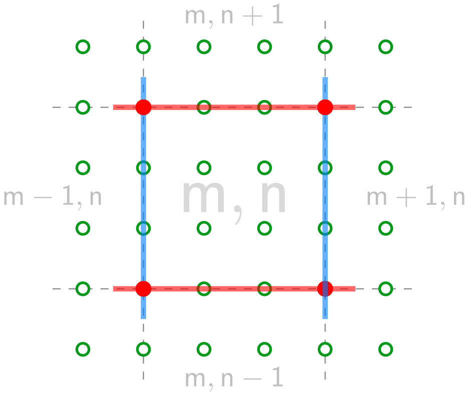
Bézier surface on one patch

$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_j(t_2)$$



Two-curves   Karcher   De Casteljau 2D

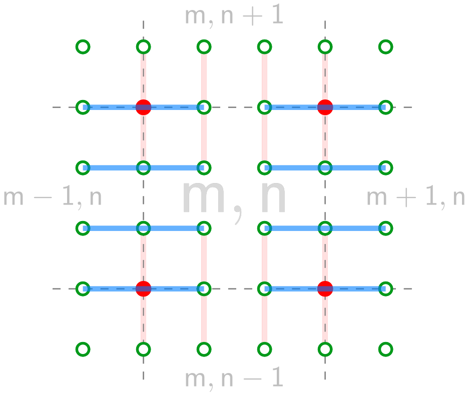
Continuity



$$b_{i,0}^{m,n} = b_{i,3}^{m,n-1}$$

$$b_{0,j}^{m,n} = b_{3,j}^{m-1,n}$$

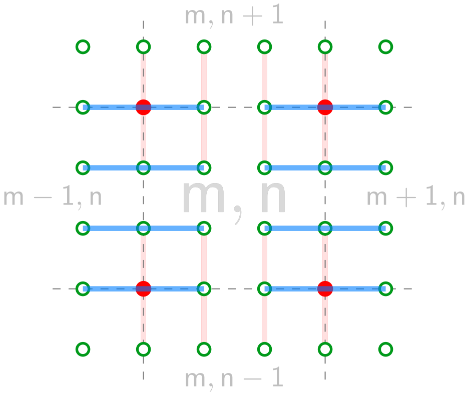
Differentiability



$$b_{0,j}^{m,n} = \frac{b_{-1,j}^{m,n} + b_{1,j}^{m,n}}{2}$$

$$b_{i,0}^{m,n} = \frac{b_{i,-1}^{m,n} + b_{i,1}^{m,n}}{2}$$

Differentiability

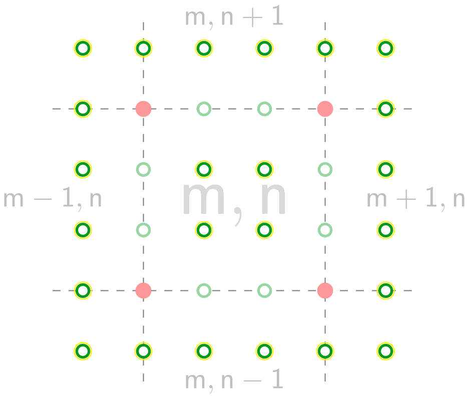


$$b_{0,j}^{m,n} = \mathrm{av}[(b_{-1,j}^{m,n}, b_{1,j}^{m,n}), (\tfrac{1}{2}, \tfrac{1}{2})]$$

$$b_{i,0}^{m,n} = \mathrm{av}[(b_{i,-1}^{m,n}, b_{i,1}^{m,n}), (\tfrac{1}{2}, \tfrac{1}{2})]$$

not sufficient...

A new definition of Bézier surfaces in $\mathcal{M}$



$$\beta_3(t_1, t_2, \mathbf{b}) = \mathrm{av}[\tilde{\mathbf{b}}, \tilde{w}_{ij}]$$

C^1 conditions!

Optimal $\mathcal{C}^1$-piecewise Bézier surface (in $\mathbb{R}^n$)

Minimization of the mean squared acceleration of the surface

In the Euclidean space...

$$\min_{b_{ij}^{mn}} \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{F}(\beta_3^{mn})$$

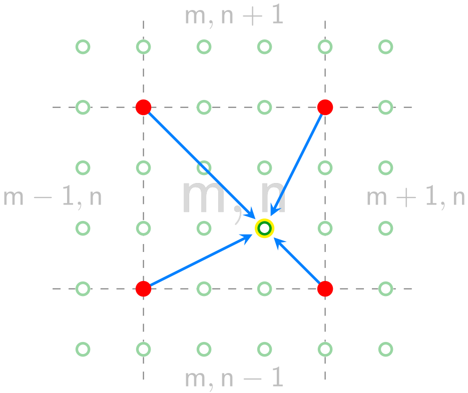
where

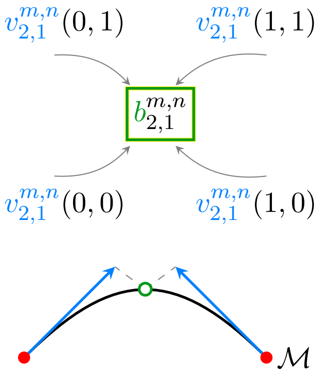
$$\hat{F}(\beta_3^{mn}) = \int_{[0,1]\times[0,1]} \left\| \frac{\partial^2 \beta_3^{mn}}{\partial(t_1, t_2)} \right\|_F^2 \mathrm{d}t_1 \mathrm{d}t_2 = \sum_{i,j,o,p=0}^{3} \alpha_{ijop}(b_{ij}^{mn} \cdot b_{op}^{mn})$$

$\ddot{B}_{o3}(t_1)$, $\ddot{B}_{p3}(t_2)$, $\ddot{B}_{i3}(t_1)$, $\ddot{B}_{j3}(t_2)$

Quadratic function, easy on the Euclidean space...
but not in $\mathcal{M}$.

Optimal surface on $\mathcal{M}$: project on tangent spaces





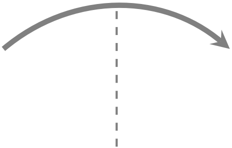
Optimal surface on manifolds

Compute $v_{i,j}^{m,n}$ on the tangent space...

Optimal surface on manifolds

... and project back to the manifold.

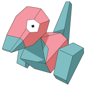
$\mathbf{Exp}_{p_{m,n}}(\cdot)$

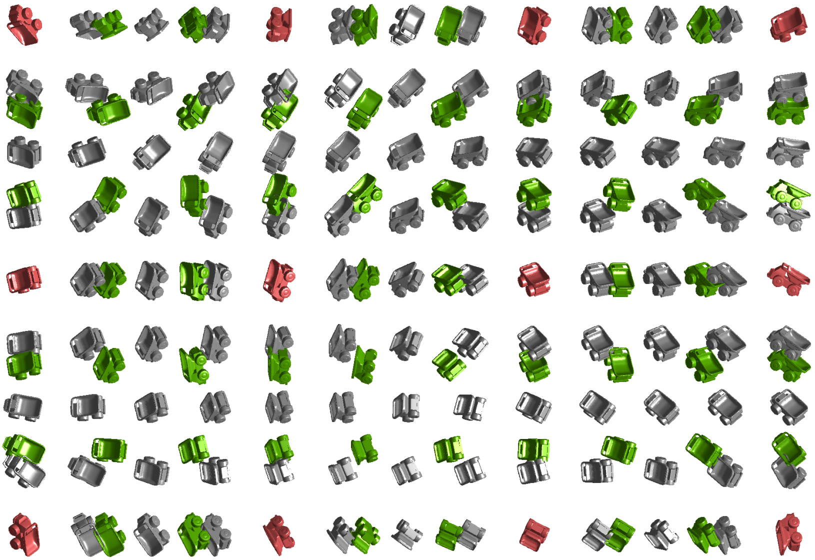


$T_{p_{m,n}}$ $v_{i,j}^{m,n}$ $b_{i,j}^{m,n}$ $\mathcal{M}$

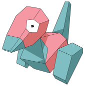
... well it's a bit more complicated ;-).

A result on SO(3)

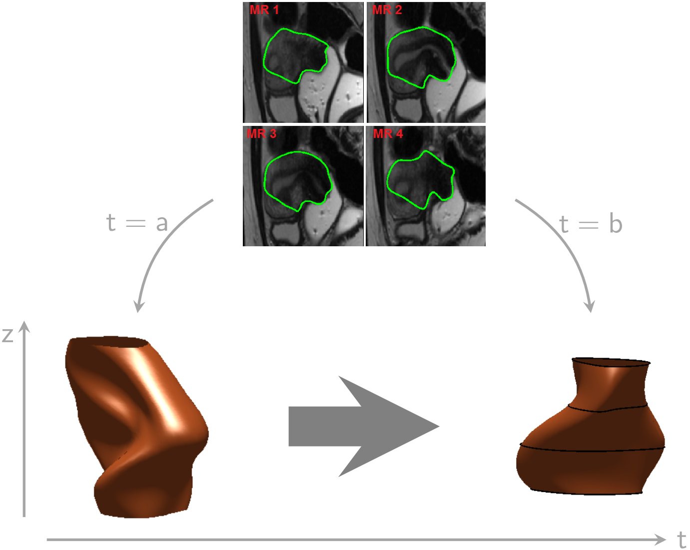




A cool result



The medical application



Interpolation with Bézier in 2D : pros and cons

✓ Optimality conditions are a closed form linear system.

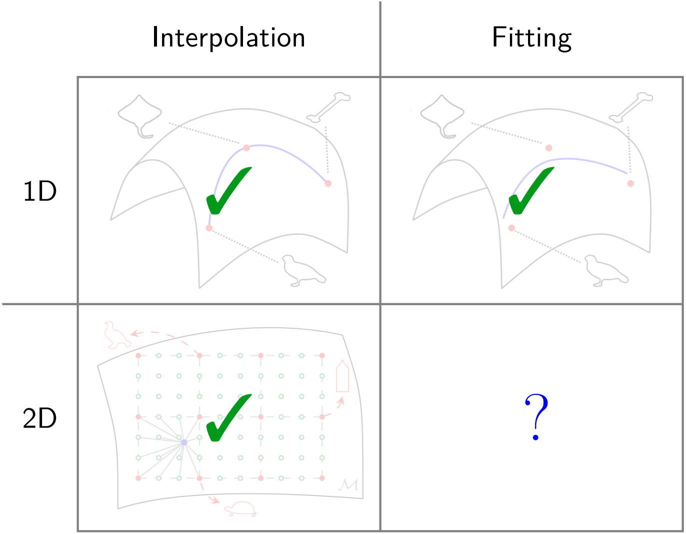
✓ Method only needs exp and log maps and parallel transport.

✓ The surface is $\mathcal{C}^1$.

✗ The control points generation might be very heavy.
Another method to generate the control points [Absil *et al.*, 2016]

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

The path...



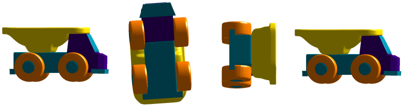
Conclusions

General C^1-**interpolative/fitting** methods on **manifolds**...
with applications in medical imaging, wind estimation, model reduction,...

light • closed form • uses few elements in $\mathcal{M}$

Summary on interpolation :
"Differentiable Piecewise-Bézier Surfaces on Riemannian Manifolds"
[Absil, Gousenbourger, Striewski, Wirth, *SIAM Journal on Imaging Sciences*, to appear].

Any questions?



Interpolation and fitting on manifolds with differentiable piecewise-Bézier functions

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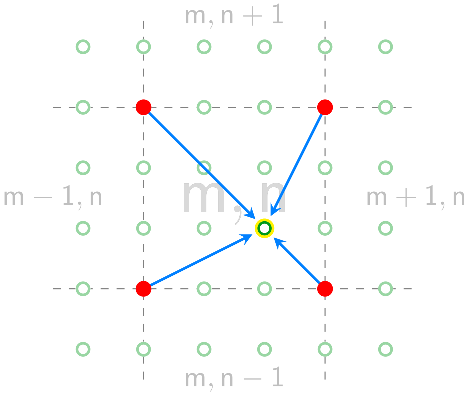
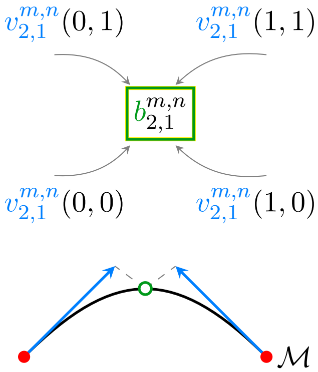
pierre-yves.gousenbourger@uclouvain.be

13 décembre 2018

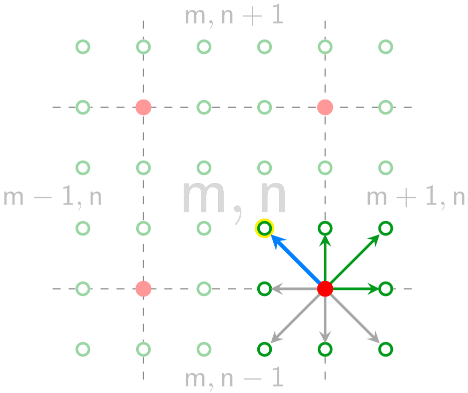
Optimal surface : prepare the manifold setting





$$\hat{F}(\beta_3^{mn}) = \sum_{i,j,o,p=0}^{3} \frac{1}{4} \alpha_{ijop} \sum_{r,s\in\{0,1\}} \left(v_{ij}^{mn}(r,s) \cdot v_{op}^{mn}(r,s) \right)$$

Optimal surface : system reduction



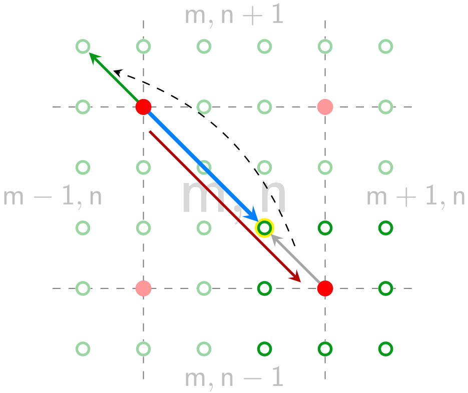
$$u_{0,1}^{m+1,n},\ u_{1,0}^{m+1,n},\ u_{1,1}^{m+1,n}$$

↓S

$$u_{-1,1}^{m+1,n} = u_{0,1}^{m+1,n} - u_{1,0}^{m+1,n}$$

C^0 and C^1 conditions

Optimal surface : constraints



$$u^{m+1,n}_{-1,-1}$$

$$\tilde{\mathrm{T}} \downarrow \mathbf{T} + \mathbf{Z}$$

$$v^{m,n}_{2,1}(0,1)$$

$$=$$

$$P_{p_{m,n+1} \leftarrow p_{m+1,n}}(u^{m+1,n}_{-1,-1})$$

$$-$$

$$\mathrm{Log}_{p_{m,n+1}}(p_{m+1,n})$$

Optimal surface : solution

The objective function $\quad L(X)_{ij} = \frac{1}{4}\sum_{o,p} \alpha_{ijop} x_{op}$

$$\min_{u_{ij}^{mn}(r',s')} \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{i,j=0}^{3} \sum_{r,s\in\{0,1\}} (L\tilde{T}SU)_{i,j,r,s}^{m,n} \cdot (\tilde{T}SU)_{i,j,r,s}^{m,n}$$

is solved through a linear system

$$U_{\text{opt}} = -(S^* T^* L T S)^{-1} (S^* T^* L Z).$$

$\tilde{\mathbf{T}} = \mathbf{T} + \mathbf{Z} \xleftarrow{\text{manifolds}} \mathbf{S} \xleftarrow{\text{constraints}} \mathbf{U}$