Differentiable Bézier interpolation on manifolds with B-splines

GAMM2017, Weimar

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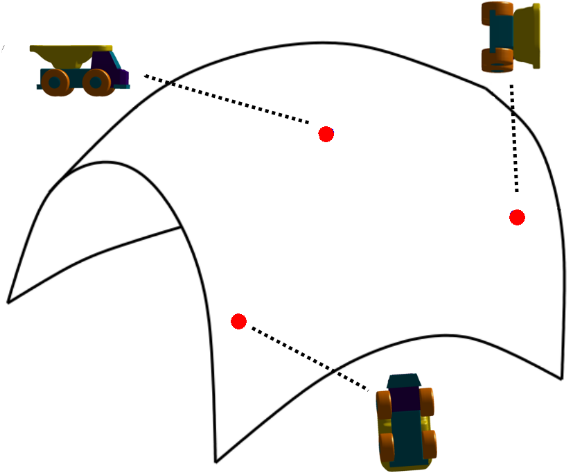
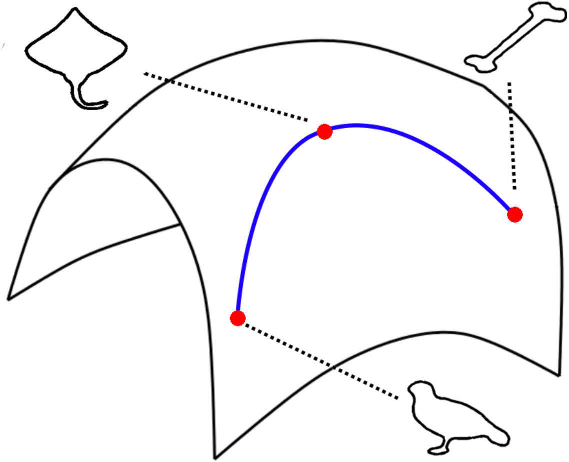
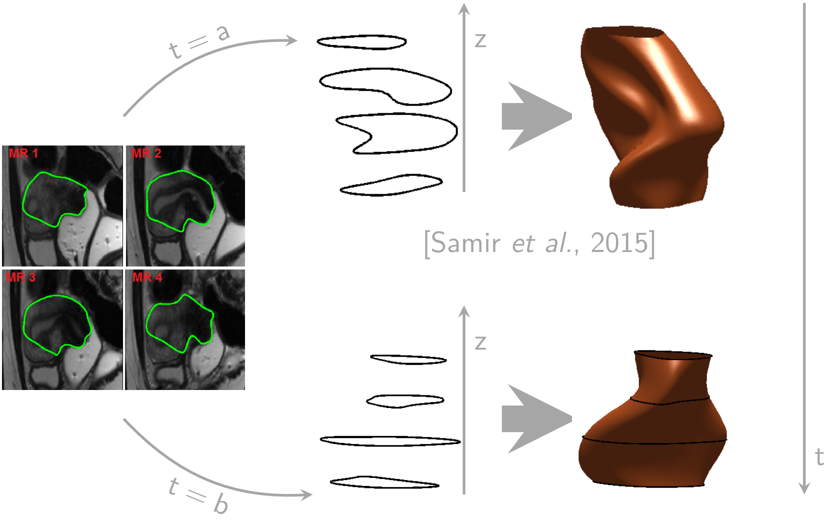
March 7, 2017

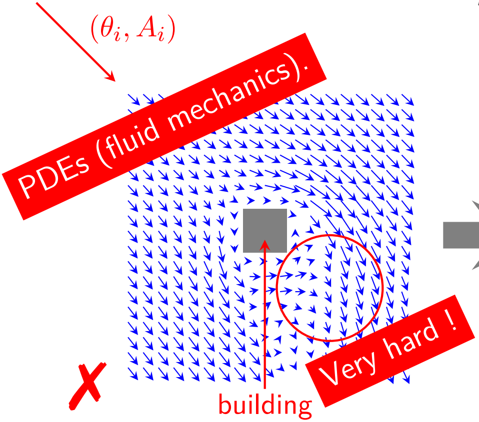




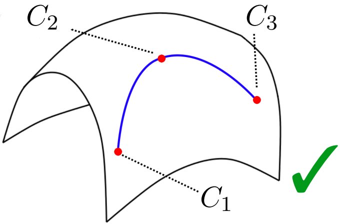
A medical application

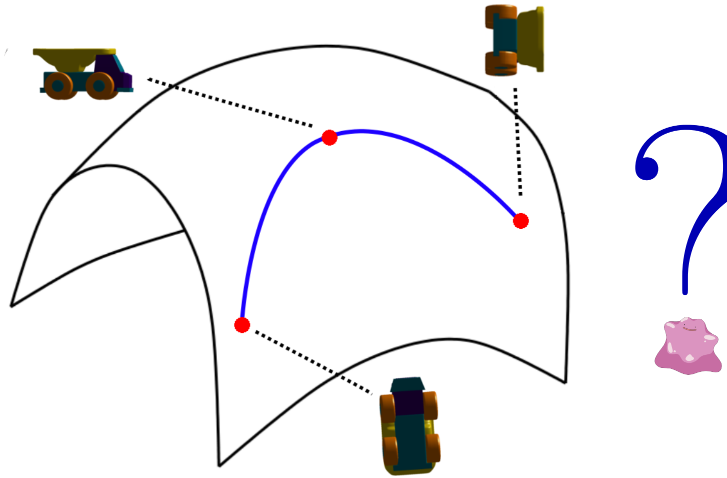


The wind field estimation



$C_i \in \mathcal{S}_+(r, n)$





How to interpolate or fit points on $\mathcal{M}$?

1.

Bézier curves

on manifolds ?

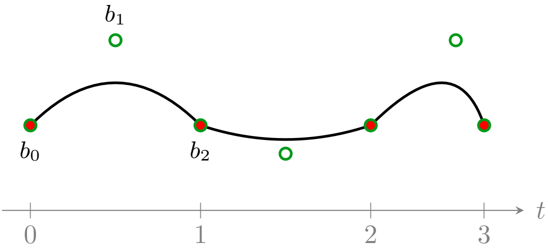
2.

How to compute

control points ?

1D : Interpolative Bézier curves

Each segment between two consecutive points is
a **Bézier curve** of degree K.

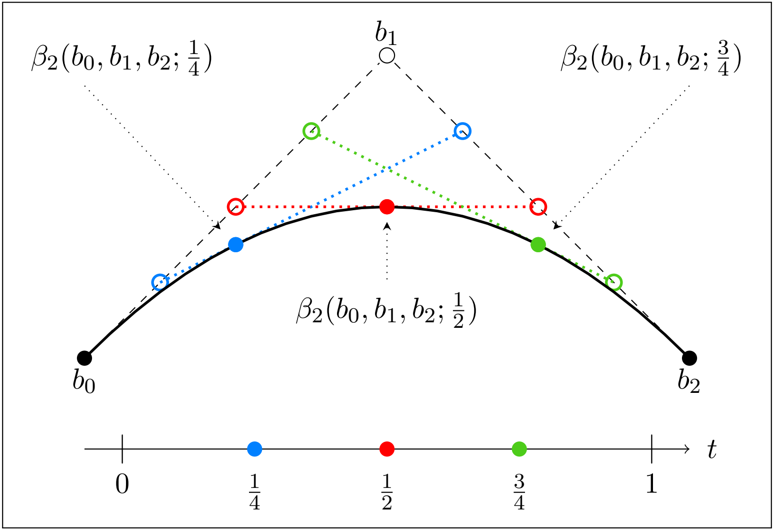


$$\mathfrak{B} = \boldsymbol{\beta}(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

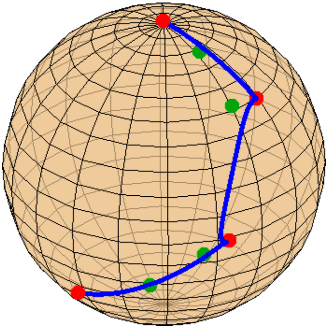
$$\beta(t, \mathbf{b}) = \sum_{i=0}^{K} b_i B_{iK}(t)$$

[G. *et al.* 2014, Arnould *et al.* 2015]

Reconstruction : the De Casteljau algorithm



Example on the sphere



It's ugly. Make it **smooth**!

1.

Bézier curves

on manifolds ?

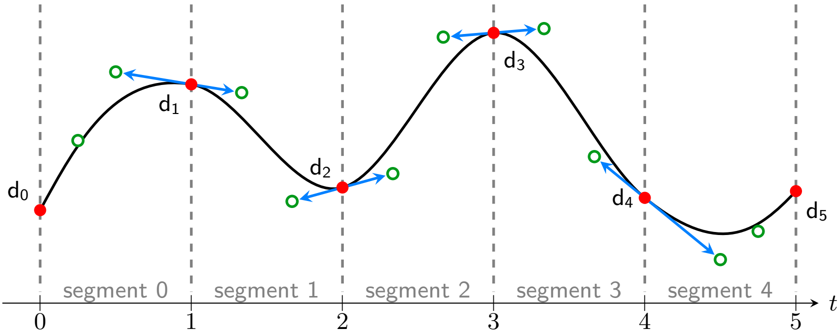
✓

2.

How to compute

control points ?

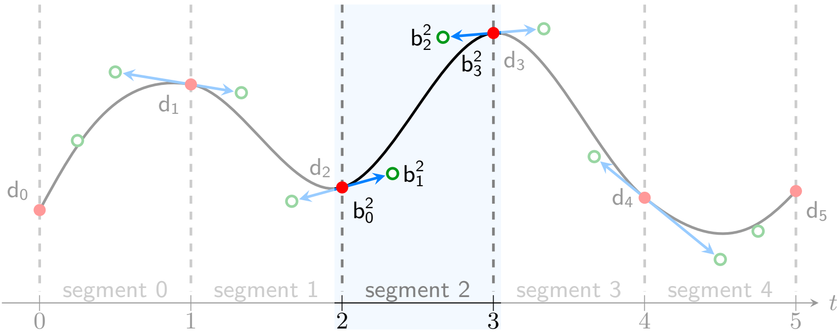
Smooth interpolation with Bézier (in $\mathbb{R}^n$) : $\mathfrak{B}(t)$



$$\mathfrak{B} = \boldsymbol{\beta}(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

Each segment is a Bézier curve smoothly connected !

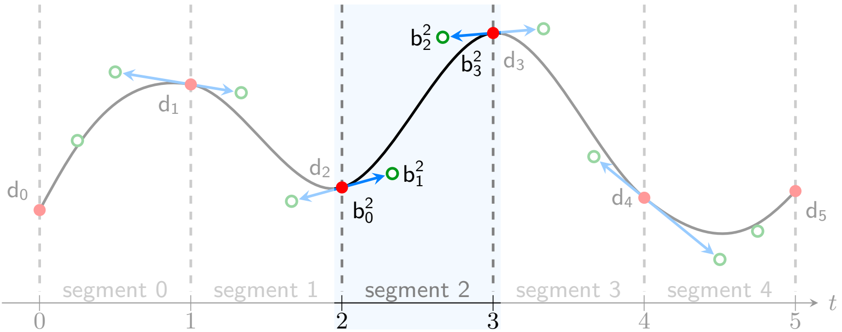
Smooth interpolation with Bézier (in $\mathbb{R}^n$) : $\mathfrak{B}(t)$



$$\mathfrak{B} = \boldsymbol{\beta}(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

Each segment is a Bézier curve smoothly connected !
Unknowns : b_i^m.

Smooth interpolation with Bézier (in $\mathbb{R}^n$) : $\mathfrak{B}(t)$



$$\mathfrak{B} = \boldsymbol{\beta}(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

Minimize the mean square acceleration $\int_0^M \|\mathfrak{B}''(t)\| \mathrm{d}t$

How to compute the control points... in $\mathbb{R}^n$

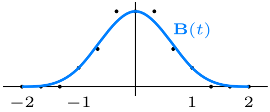
In $\mathbb{R}^n$

Unique $\mathcal{C}^2$-interpolating piecewise-cubic Bézier curve

second derivative
vanishes
at the boundaries

$$\min_{b_i^m} \int_0^M \|\mathfrak{B}''(t)\| \mathrm{d}t$$

From the B-spline to the control points



$$\mathfrak{B} := \sum_{m=-1}^{M+1} \alpha_m \mathbf{B}(t-m)$$

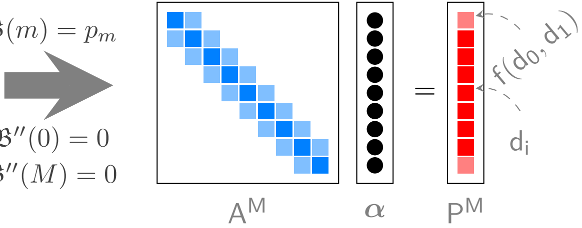
[Farin, 2002]

Under interpolation constraints,
we find the B-spline coefficients α_m.

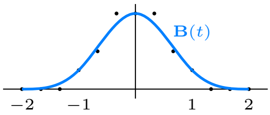
$\mathfrak{B}(m) = p_m$

$\mathfrak{B}''(0) = 0$

$\mathfrak{B}''(M) = 0$

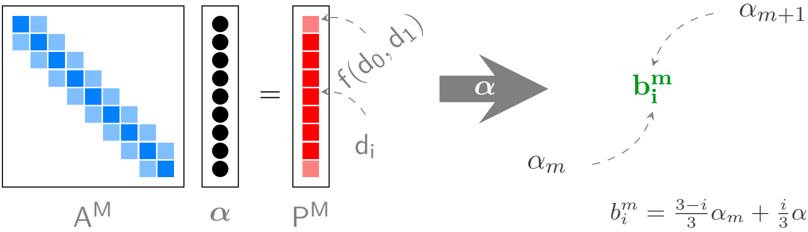


From the B-spline to the control points



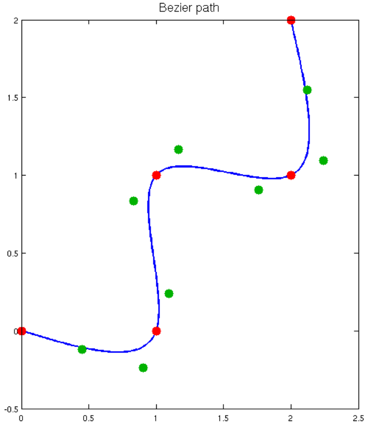
$$\mathfrak{B} = \boldsymbol{\beta}(t - m, \mathbf{b}^m), \quad m = \lfloor t \rfloor$$

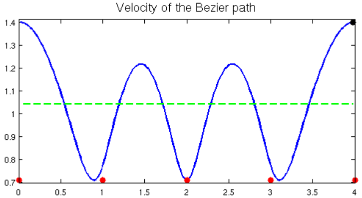
The control points $\mathbf{b}^m$ are convex combinations of α_m.

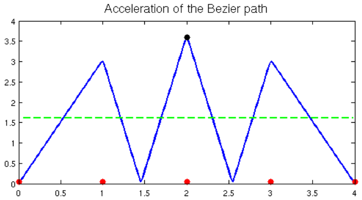


$b_i^m = \frac{3-i}{3}\alpha_m + \frac{i}{3}\alpha_{m+1}$

A result on $\mathbb{R}^2$







Optimal $\mathcal{C}^1$-piecewise Bézier interpolation (on $\mathcal{M}$)

- The control points are given by :

$$b_i^m = \frac{3-i}{3}\alpha_m + \frac{i}{3}\alpha_{m+1} = \sum_{j=0}^{n} q_{i,j} d_j$$

- These points are invariant under translation, *i.e.* :

$$b_i^m - d^{ref} = \sum_{j=0}^{n} q_{i,j}(d_j - d^{ref})$$

- On manifolds : projection to the **tangent space** of d^{ref} with the **Log**, as $a - b \Leftrightarrow \mathrm{Log}_b(a)$

$$v_i = \mathrm{Log}_{d^{ref}}(b_i^m) = \sum_{j=0}^{n} q_{i,j} \mathrm{Log}_{d^{ref}}(d_j)$$

- Back to the manifold with the **Exp** : $b_i^m = \mathrm{Exp}_{d^{ref}}(v_i)$.

1.

Bézier curves

on manifolds ?

✓

2.

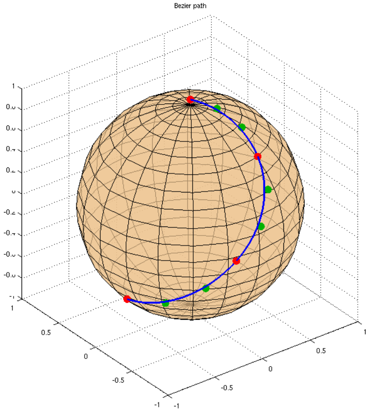
How to compute

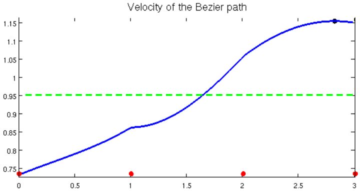
control points ?

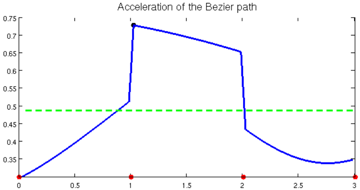
✓

Results (and bonus ?)

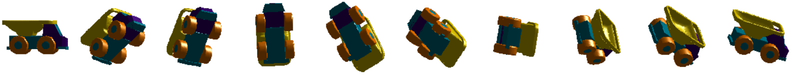
Result on the sphere

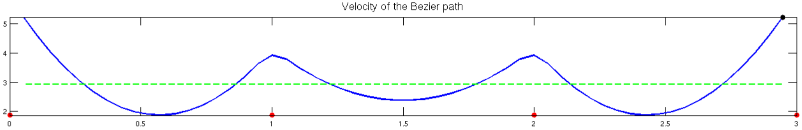


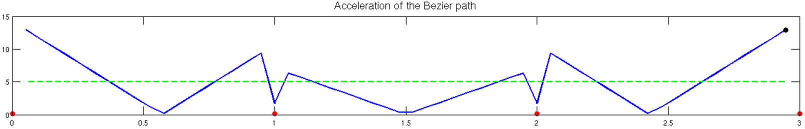




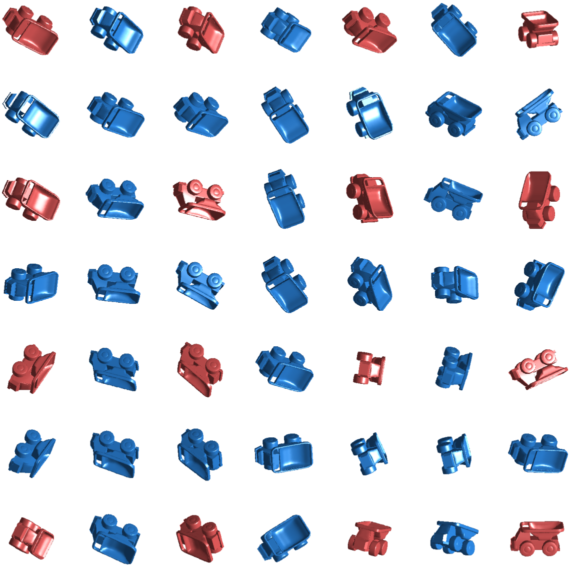
Result on SO(3)







In 2D on SO(3) ?



Interpolation with Bézier : pros and cons

✓ Optimality conditions are a closed form linear system.

✓ Method only needs exp and log maps.

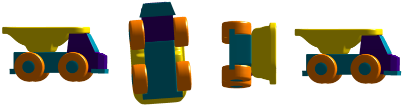
✓ The curve is $\mathcal{C}^1$.

✗ No guarantee on the optimality when $\mathcal{M}$ is not flat.

[G. *et al.*, 2016] [Arnould *et al.*, 2015] [Pyta *et al.*, 2016]

Summary on interpolation
"Differentiable Piecewise-Bézier Surfaces on Riemannian Manifolds"
[Absil, Gousenbourger, Striewski, Wirth, *SIAM Journal on Imaging Sciences*, 2017].

Any questions?



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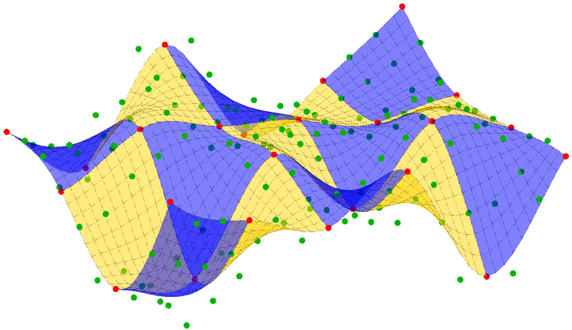
From 1D to 2D ?

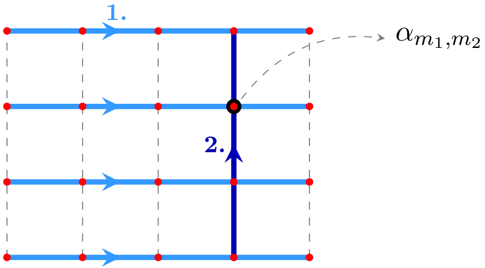
Each patch between four neighbour points is
a **Bézier surface** of degree K.



$$\mathfrak{B}(t_1, t_2) = \boldsymbol{\beta}(t_1 - m_1, t_2 - m_2, \mathbf{b}), \text{ with } m_1 = \lfloor t_1 \rfloor \text{ and } m_2 = \lfloor t_2 \rfloor$$

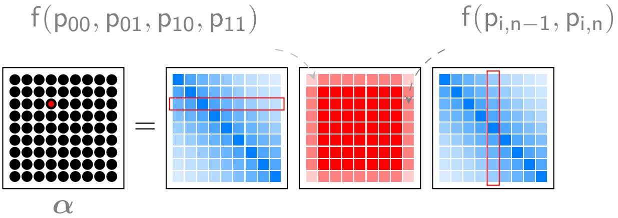
From 1D to 2D ? (in $\mathbb{R}^n$)

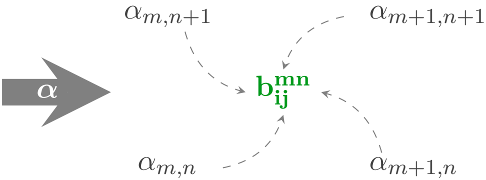
The surface $\mathfrak{B}$ is a tensorized version of the curve in step 1.
We use step 1 in direction t_1 then t_2 to obtain the coefficients α_{mn}.



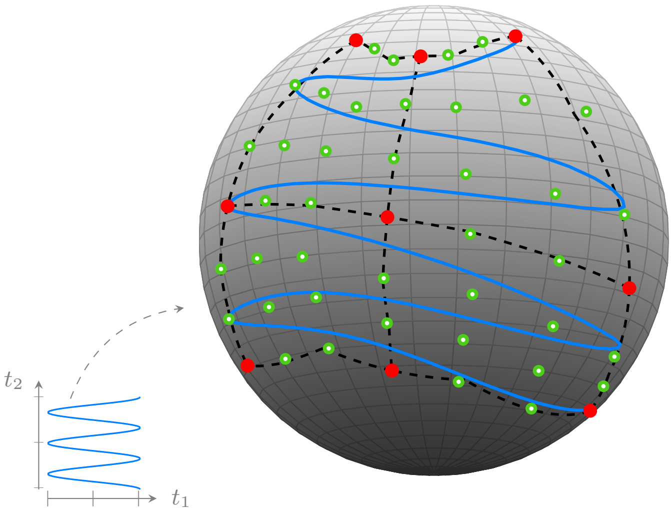
From 1D to 2D ? (in $\mathbb{R}^n$)

The control points **b** are convex combinations of α_{mn}.





Result in 2D ?



$$S = \{(t_1, t_2) : t_1 = 1 + \cos(3\pi t_2)\}$$

In 2D on SO(3) ?