Curve fitting on manifolds with Bézier and blended curves

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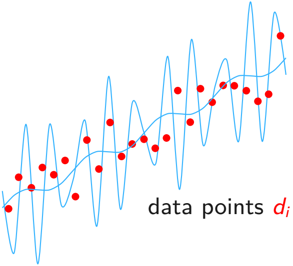
What is the problem?

Find a $\mathcal{C}^1$ curve $\mathbf{B}(t)$, s.t.



Bézier curve!

Bézier spline!

$$\underset{\mathbf{B}\in\Gamma}{\operatorname{argmin}}\, E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\mathbf{B}(t)}{\mathrm{d}t^2} \right\|^2_{\mathbf{B}(t)} \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$

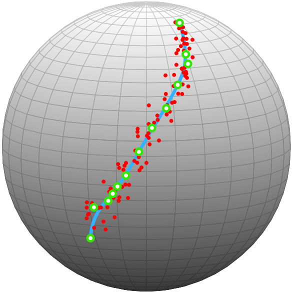
regularizer

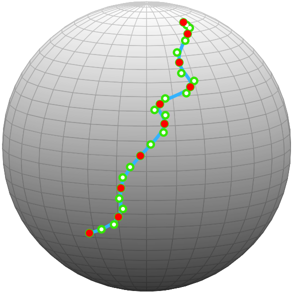
data attachment

Data points $d_i \in \mathbb{R}^2$

curve $\mathbf{B} : [0, n] \to \mathbb{R}^2$

Why is this important? – Sphere



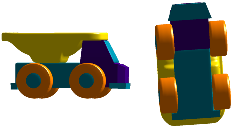


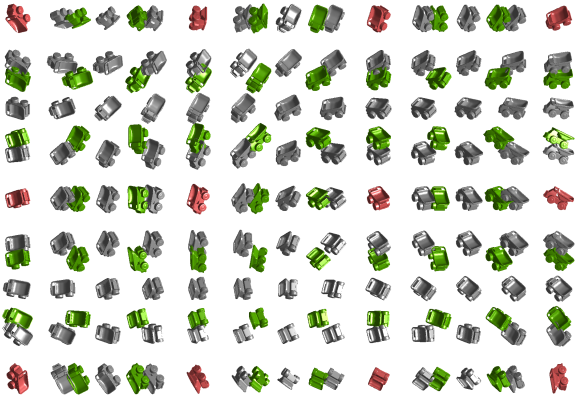
storm trajectories
birds migrations
distress planes roadmaps extrapolation

Data points $d_i \in \mathbb{S}^2$

curve $\mathbf{B} : [0, n] \to \mathbb{S}^2$

Why is this important? – Orthogonal group





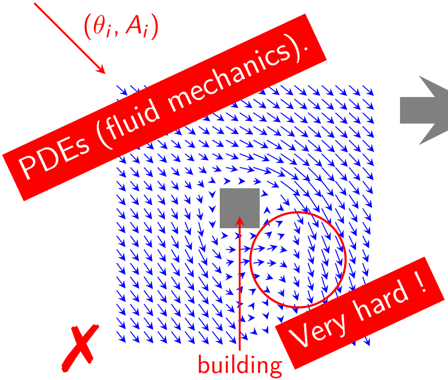
Rigid rotations of 3D objects
3D printing plannings

Computer vision, video games

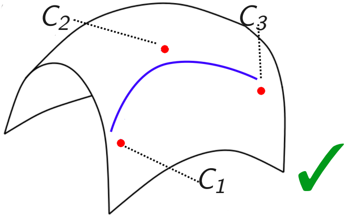
Data points $d_i \in \mathrm{SO}(3)$

curve $\mathbf{B} : [0, n] \to \mathrm{SO}(3)$

Why is this important? - SDP matrices of size p,rank r



$C_i \in \mathcal{S}_+(p,r)$

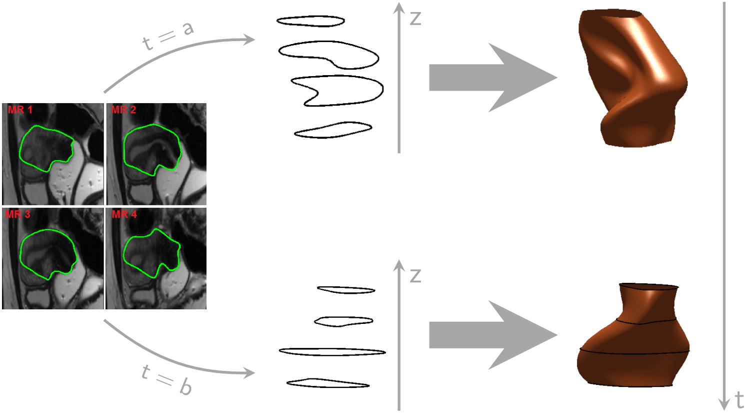


Wind field estimation for UAV

Data points $d_i \in \mathcal{S}_+(p,r)$

curve **B** : $[0,n] \to \mathcal{S}_+(p,r)$

Why is this important? - Shape space



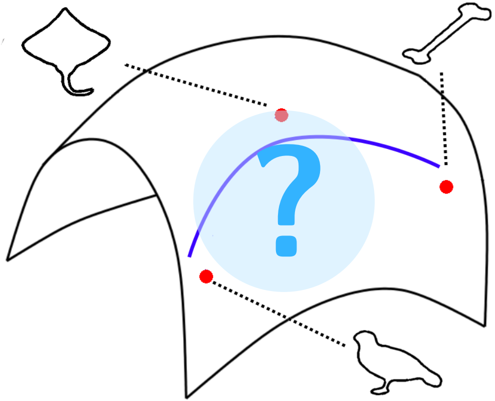
medical imaging, harmed soldiers rehab'

Data points $d_i \in \mathcal{S}$

curve **B** : $[0, n] \to \mathcal{S}$

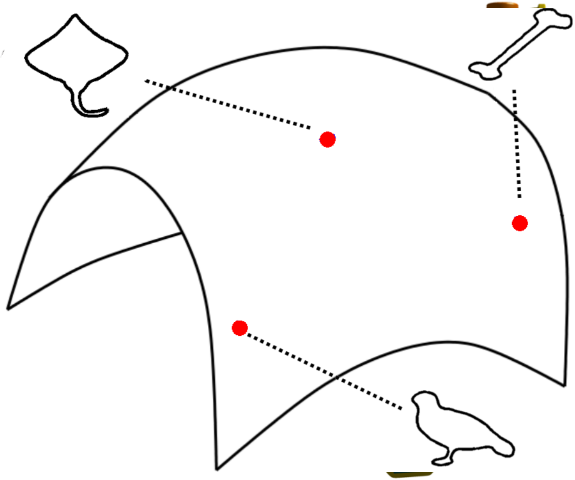
What they have in common

$\mathbb{S}^2$, SO(3), $\mathcal{S}_+(p, r)$, $\mathcal{S}$,... are Riemannian manifolds.

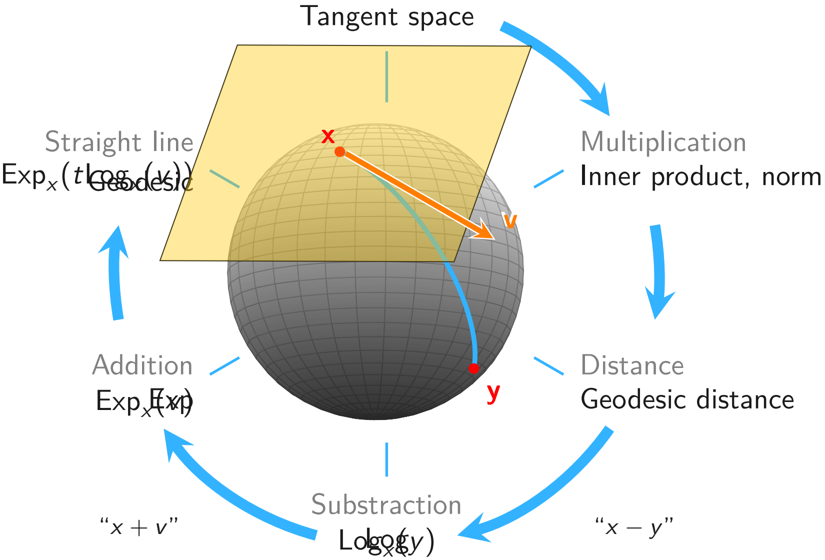


- Fast
- low complexity
- meaningful
- easy to use

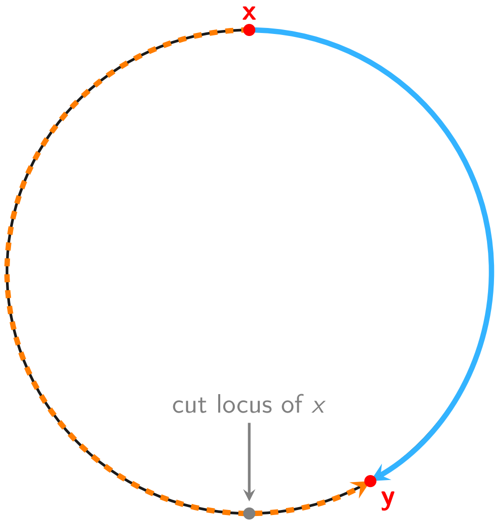
What is a manifold?



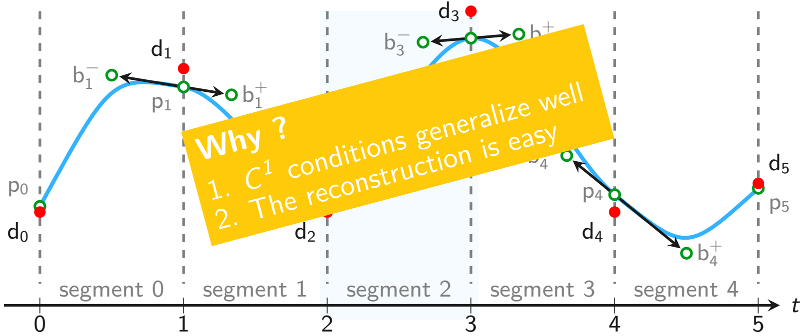
The sphere as an example



The curse of the curvature: the cut locus



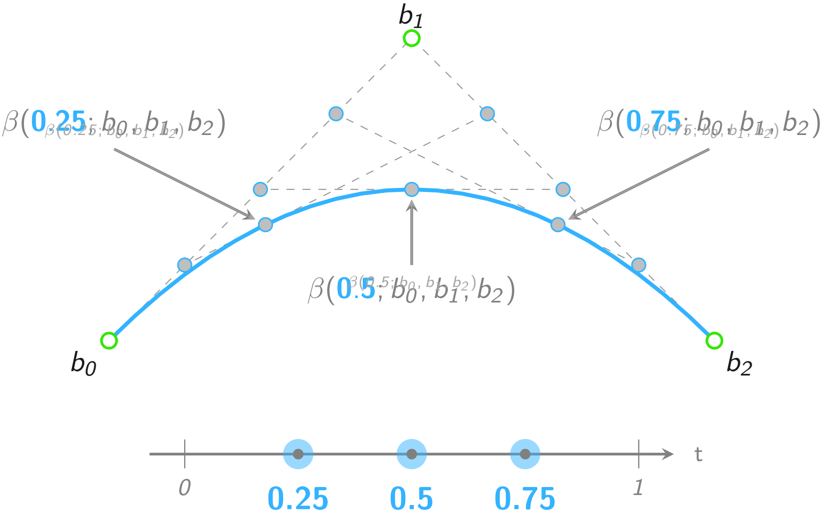
$\mathbf{B}(t)$ is a piecewise cubic Bézier curve



$$\mathbf{B} = \beta(t - m, \mathbf{b}^m) \text{ with } m = \lfloor t \rfloor$$

Each segment is a Bézier curve smoothly connected!
Unknowns: b_i^+, b_i^-, p_i.

Why Bézier? – De Casteljau Algorithm generalizes well



How to compute the control points?

$$\boxed{\text{in } \mathbb{R}^d}$$

Unique $\mathcal{C}^2$ smoothing polynomial spline

$$\text{s.t. } \min_{b_i^m} \int_0^M \|\mathbf{B}''(t)\| \mathrm{d}t$$

(Long story short)

$$b_i^m = \sum_{j=0}^{n} q_{i,j} d_j$$

Generalization to $\mathcal{M}$

$$\boxed{\text{in } \mathcal{M} \text{ ?}}$$

1. Invariance to translation to a point d_{ref}.
2. Translation to d_{ref} is a Riemannian Log on $\mathbb{R}^r$.
3. Exponentiall map to go back to $\mathcal{M}$.
4. Compute p_i with the manifold-valued $\mathcal{C}^1$ condition.

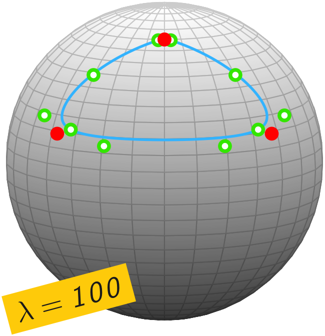
$$\log_{d_{\text{ref}}}(p_i^m) = \sum_{j=0}^{n} q_{i,j} \log_{d_{\text{ref}}}(d_j)$$

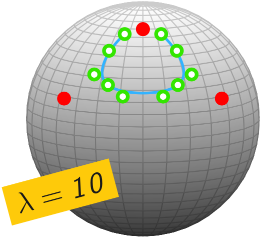
$$p_i^m = \exp_{d_{\text{ref}}}\left(\sum_{j=0}^{n} q_{i,j} \log_{d_{\text{ref}}}(d_j)\right)$$

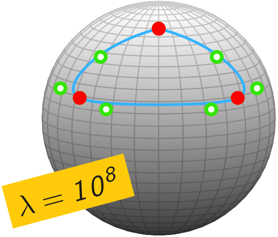
Generalization to $\mathcal{M}$ - illustration

A drawing is worth 10 explanations... ;-)

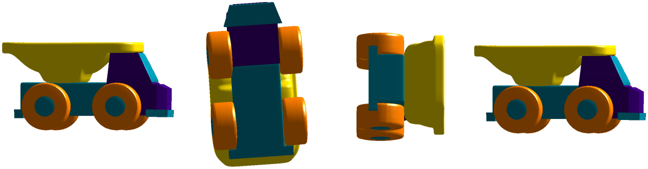
Results show that it works... ($\mathbb{S}^2$)



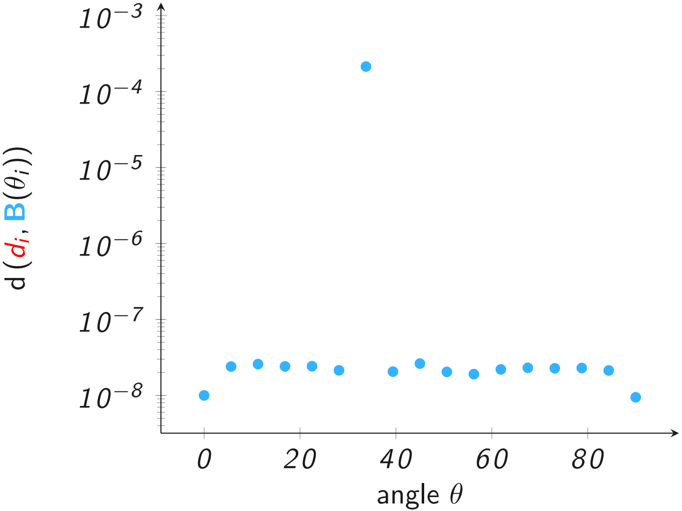




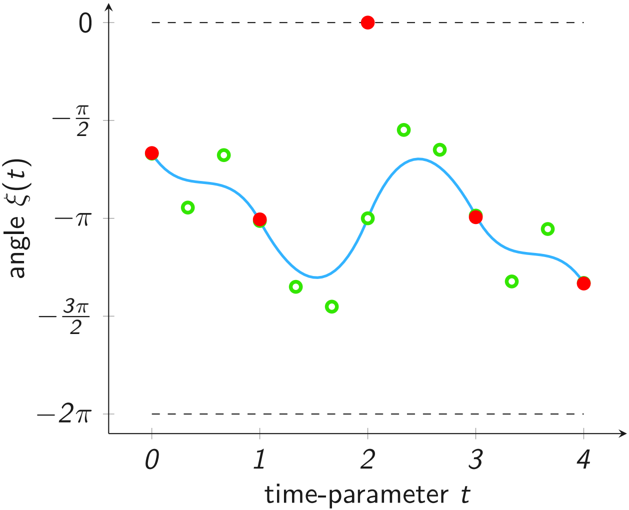
Results show that it works... (SO(3))



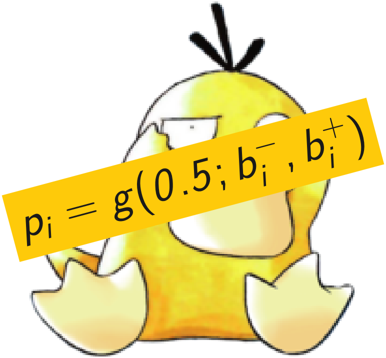
... but it actually fails sometimes ($\mathcal{S}_+(p,r)$)



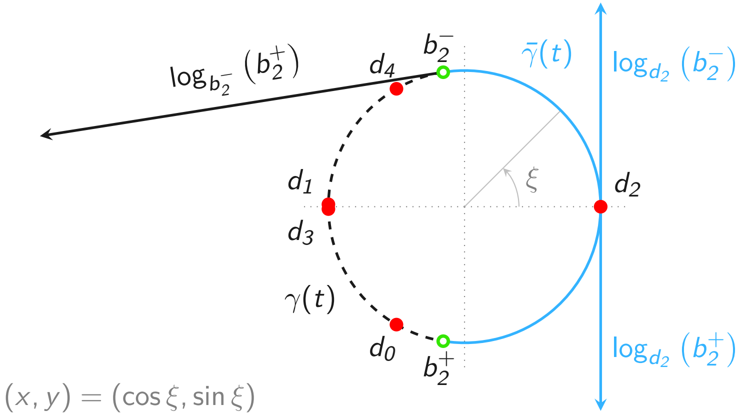
... but it actually fails sometimes ($\mathbb{S}^1$)



So what's wrong?



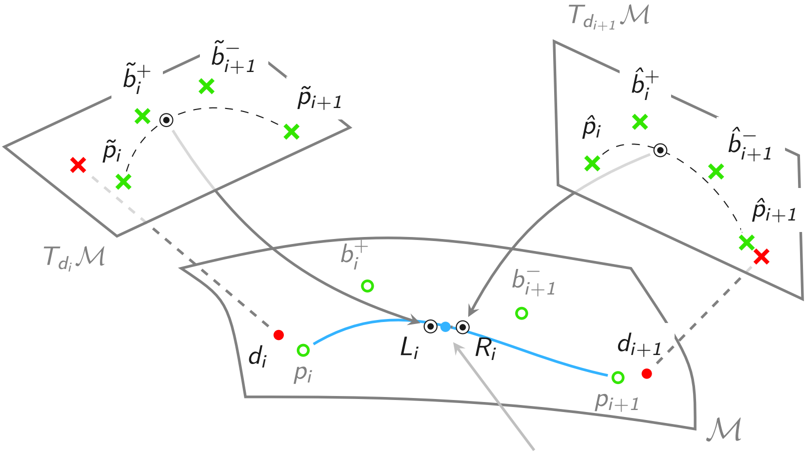
The failure revealed! ($\mathbb{S}^1$)



What solutions?

A drawing is worth 10 explanations...

The Cubic Blended Splines Algorithm



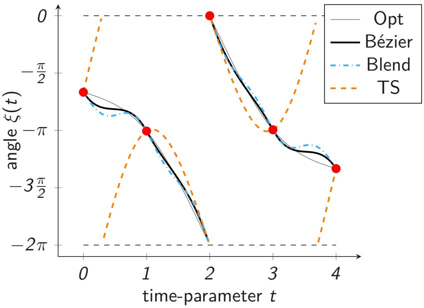
$\mathbf{B}(t) = \mathrm{av}[(L_i, R_i), (1-w, w)]$

is a weighted geodesic averaging of L_i and R_i

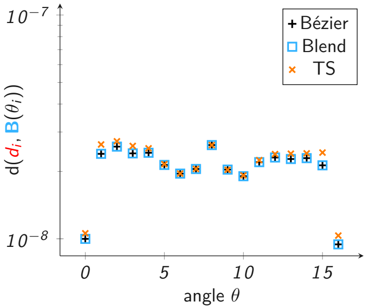
with a weight $w(t) = 3t^2 - 2t^3$

Results: all is well!

Angle of the Bézier-like curve on $\mathbb{S}^1$

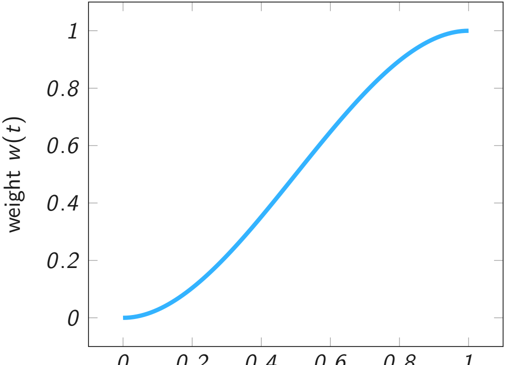


Interpolation error on the wind field data



Properties and take-home message

1. $\mathbf{B}(t_i) = d_i$ when $\lambda \to \infty$;
2. $\mathbf{B}(t)$ is $\mathcal{C}^1$; $(\beta_i(t) = \mathrm{av}[(L_i, R_i), (1 - w(t), w(t))])$
3. $\mathbf{B}(t)$ is the natural smoothing spline when $\mathcal{M} = \mathbb{R}^r$
4. $\mathbf{B}(t_i) = d_i$ when $\lambda \to \infty$;
5. $\mathbf{B}(t)$ is $\mathcal{C}^1$;
6. $\mathbf{B}(t)$ is the natural smoothing spline when $\mathcal{M} = \mathbb{R}^r$



Limitations

The objective was., for $\{d_i\}_{i=0}^{n} \in \mathcal{M}$..

$$\underset{\mathbf{B}\in\Gamma}{\operatorname{argmin}}\, E_\lambda(\mathbf{B}) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2 \mathbf{B}(t)}{\mathrm{d}t^2} \right\|_{\mathbf{B}(t)}^2 \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\mathbf{B}(t_i), d_i),$$

regularizer

data attachment

Are the 6 properties always true? The cut locus is still a curse.

Is it **far** from the optimal **B**?

- Yes if the points are spread out!
- No otherwise.

Future work

Is it **far** from the optimal **B**?

- Ongoing work with R.Bergmann
- No theoretical bound yet.

Generalization to **2D**, **3D**, is open.

Application to real data is awaiting.

This is the end of the talk :-)

Curve fitting on manifolds with Bezier and blended curve

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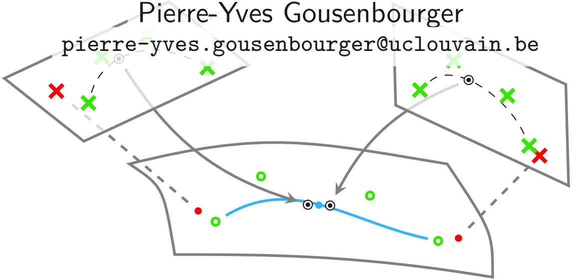

P.Y.G, E. Massart, P.-A. Absil, ***Data fitting on manifolds with composite Bézier-like curves and blended cubic splines***, JMIV (MIA2018) - under review.