Interpolation on manifolds with differentiable surfaces of Bézier

Benelux Meeting 2016

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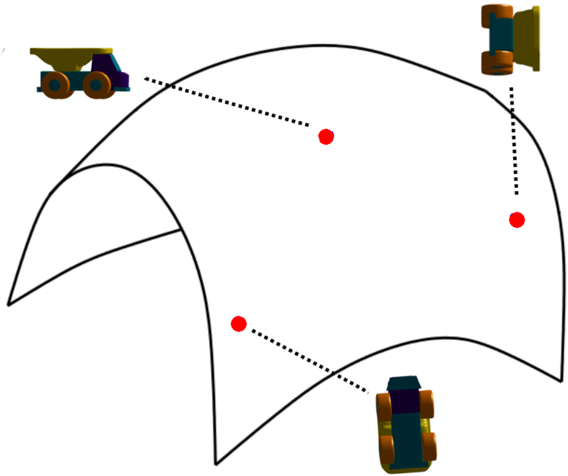
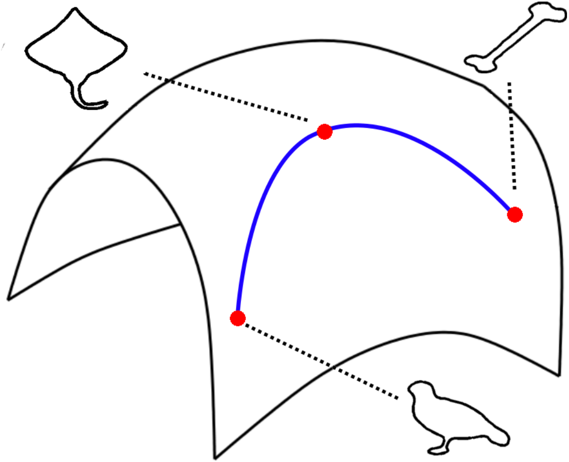
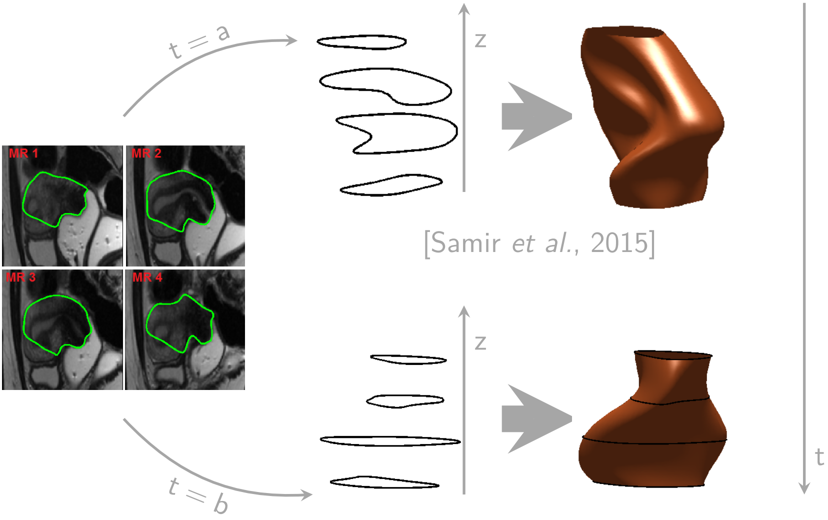




Some medical application (it's the topic, isn't it ?)



[Samir *et al.*, 2015]

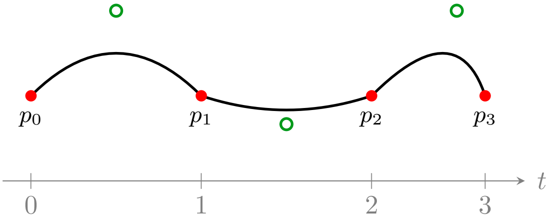
?

How to interpolate points on manifolds...

... in 2D ?

1D : Interpolative Bézier curves

Each segment between two consecutive points is
a **Bézier curve** of degree K.

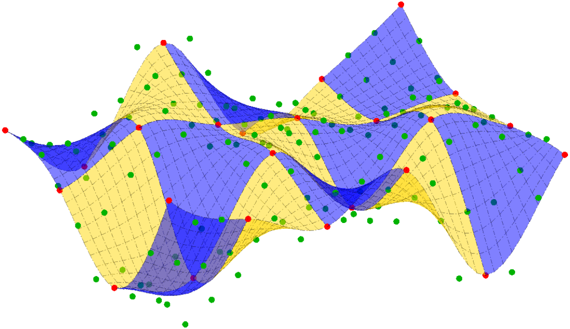


$$\beta_K(t, \mathbf{b}) = \sum_{i=0}^{K} b_i B_{iK}(t)$$

[G. *et al.* 2014, Arnould *et al.* 2015]

2D : Interpolative Bézier surface

Each patch between four neighbour points is
a **Bézier surface** of degree K.

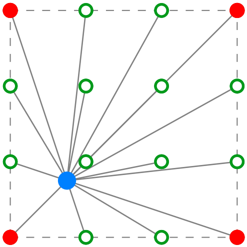


$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} B_{iK}(t_1) B_{jK}(t_2)$$

One patch

Bézier surfaces

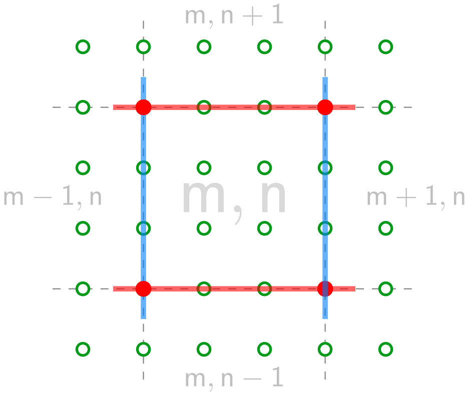
$$\beta_K(t_1, t_2, \mathbf{b}) = \sum_{i=0}^{K} \sum_{j=0}^{K} b_{ij} \overbrace{B_{iK}(t_1) B_{jK}(t_2)}^{w_{ij}} = \mathrm{av}[\mathbf{b}, w_{ij}]$$



Karcher

Many patches

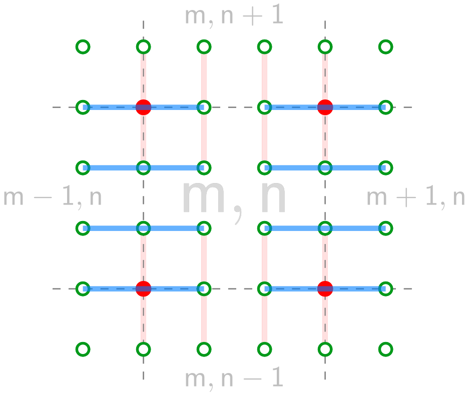
Continuity ?



$b_{i,0}^{m,n} = b_{i,3}^{m,n-1}$

$b_{0,j}^{m,n} = b_{3,j}^{m-1,n}$

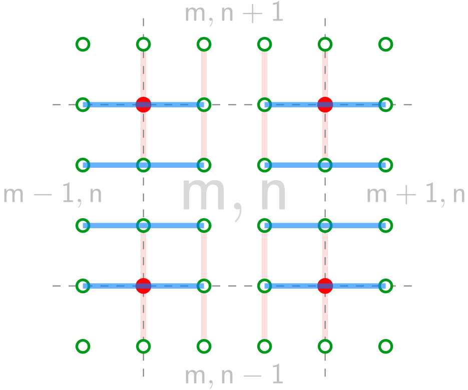
Differentiability ?



$$b_{0,j}^{m,n} = \frac{b_{-1,j}^{m,n} + b_{1,j}^{m,n}}{2}$$

$$b_{i,0}^{m,n} = \frac{b_{i,-1}^{m,n} + b_{i,1}^{m,n}}{2}$$

Differentiability ?

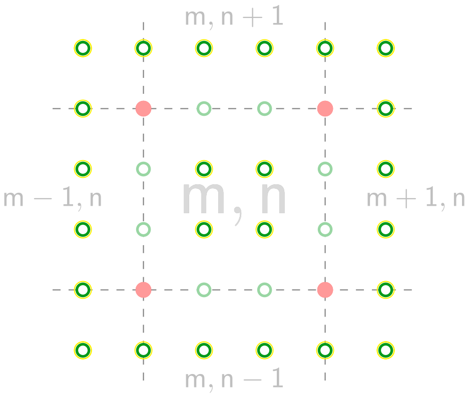


$$b^{m,n}_{0,j} = \mathrm{av}[(b^{m,n}_{-1,j}, b^{m,n}_{1,j}), (\tfrac{1}{2}, \tfrac{1}{2})]$$

$$b^{m,n}_{i,0} = \mathrm{av}[(b^{m,n}_{i,-1}, b^{m,n}_{i,1}), (\tfrac{1}{2}, \tfrac{1}{2})]$$

not sufficient...

A new definition of Bézier surfaces in $\mathcal{M}$



$$\beta_3(t_1, t_2, \mathbf{b}) = \mathrm{av}[\tilde{\mathbf{b}}, \tilde{w}_{ij}]$$

C^1 conditions!

Which control points ?

Optimal surface : objective function

In the Euclidean space...

$$\min_{b_{ij}^{mn}} \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{F}(\beta_3^{mn})$$

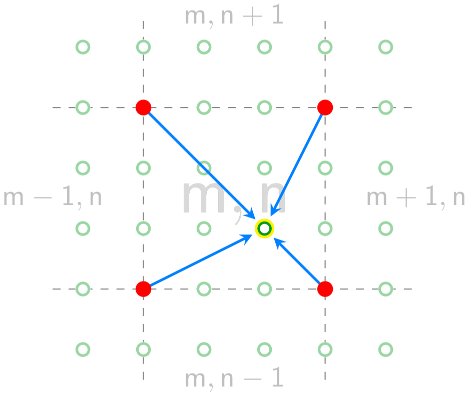
where

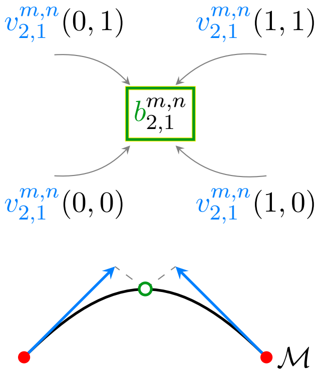
$$\hat{F}(\beta_3^{mn}) = \int_{[0,1]\times[0,1]} \left\| \frac{\partial^2 \beta_3^{mn}}{\partial(t_1, t_2)} \right\|_F^2 \mathrm{d}t_1 \mathrm{d}t_2 = \sum_{i,j,o,p=0}^{3} \alpha_{ijop} (b_{ij}^{mn} \cdot b_{op}^{mn})$$

$\ddot{B}_{o3}(t_1)$, $\ddot{B}_{p3}(t_2)$, $\ddot{B}_{i3}(t_1)$, $\ddot{B}_{j3}(t_2)$ → α_{ijop}

Quadratic function, easy on the Euclidean space...
but not in $\mathcal{M}$.

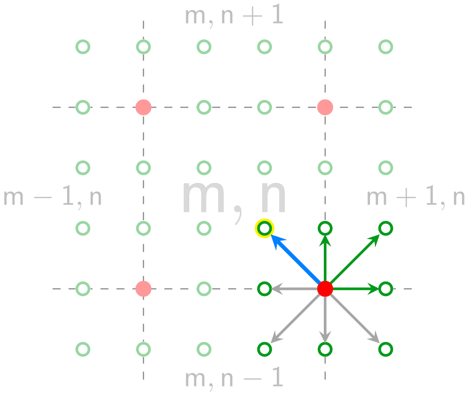
Optimal surface : prepare the manifold setting





$$\hat{F}(\beta_3^{mn}) = \sum_{i,j,o,p=0}^{3} \frac{1}{4} \alpha_{ijop} \sum_{r,s\in\{0,1\}} \left(v_{ij}^{mn}(r,s) \cdot v_{op}^{mn}(r,s) \right)$$

Optimal surface : system reduction



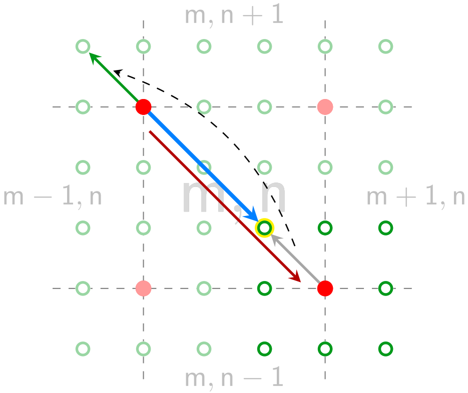
$$u_{0,1}^{m+1,n},\ u_{1,0}^{m+1,n},\ u_{1,1}^{m+1,n}$$

↓ S

$$u_{-1,1}^{m+1,n} = u_{0,1}^{m+1,n} - u_{1,0}^{m+1,n}$$

C^0 and C^1 conditions

Optimal surface : constraints



$$u^{m+1,n}_{-1,-1}$$

$$\tilde{\mathrm{T}} \downarrow \mathbf{T} + \mathbf{Z}$$

$$v^{m,n}_{2,1}(0,1)$$

$$=$$

$$P_{p_{m,n+1} \leftarrow p_{m+1,n}}(u^{m+1,n}_{-1,-1})$$

$$-$$

$$\mathrm{Log}_{p_{m,n+1}}(p_{m+1,n})$$

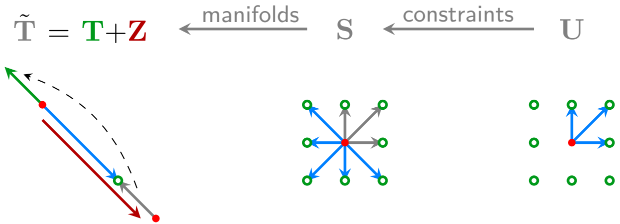
Optimal surface : solution

The objective function $L(X)_{ij} = \frac{1}{4}\sum_{o,p}\alpha_{ijop}x_{op}$

$$\min_{u_{ij}^{mn}(r',s')} \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{i,j=0}^{3}\sum_{r,s\in\{0,1\}} (L\tilde{T}SU)_{i,j,r,s}^{m,n} \cdot (\tilde{T}SU)_{i,j,r,s}^{m,n}$$

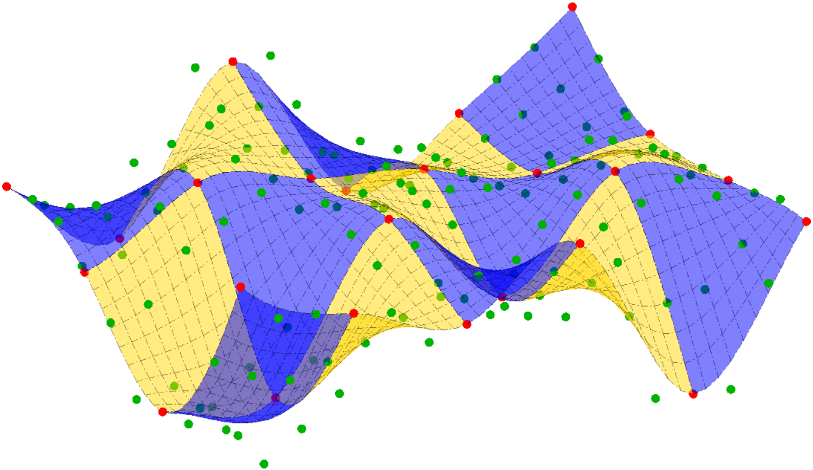
is solved through a linear system

$$U_{\text{opt}} = -(S^*T^*LTS)^{-1}(S^*T^*LZ).$$

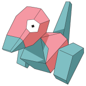


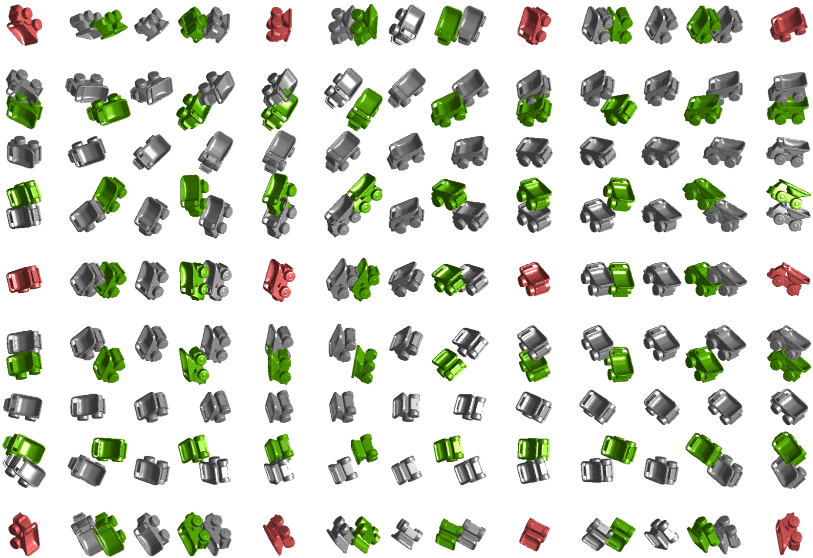
Results

A result on $\mathbb{R}^2$

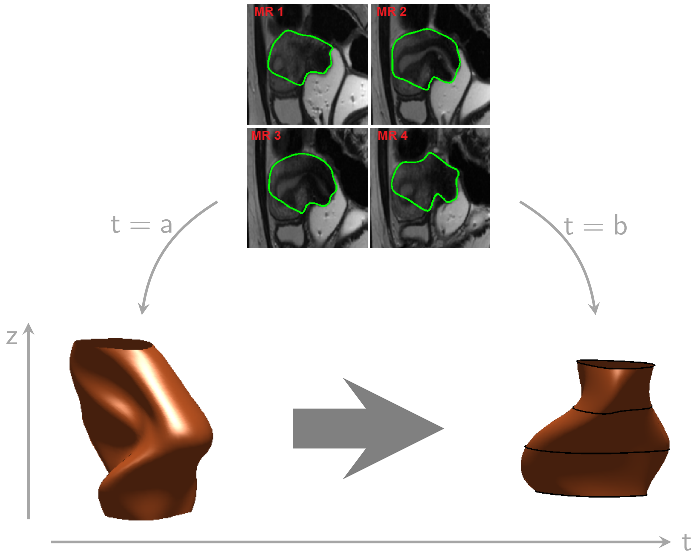


A result on SO(3)





The medical application



Conclusions

General C^1**-interpolative** method on **manifolds**...
with applications in medical imaging.

light • reduces the dimension • general

Faster method with controlled error ? Soon in ESANN2016.

Any questions ?

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