# Differentiable piecewise-Bézier interpolation on Riemannian manifolds

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## Goal of the game

**Interpolate data points $p_{mn}$ on a manifold $\mathcal{M}$ with a smooth ($\mathcal{C}^1$) piecewise Bézier surface**

$\mathfrak{B} : [0,M] \times [0,N] \to \mathcal{M} : (t_1,t_2) \mapsto \beta(t_1 - m, t_2 - n, \mathbf{b}^{mn})$

with $m = \lfloor t_1 \rfloor$ and $n = \lfloor t_2 \rfloor$

**Unknowns:** 12 control points $\mathbf{b}^{mn} = \{b_{ij}^{mn}\} \subset \mathcal{M}$ per *patch*.

**Tool:** cubic Bézier surfaces patched together

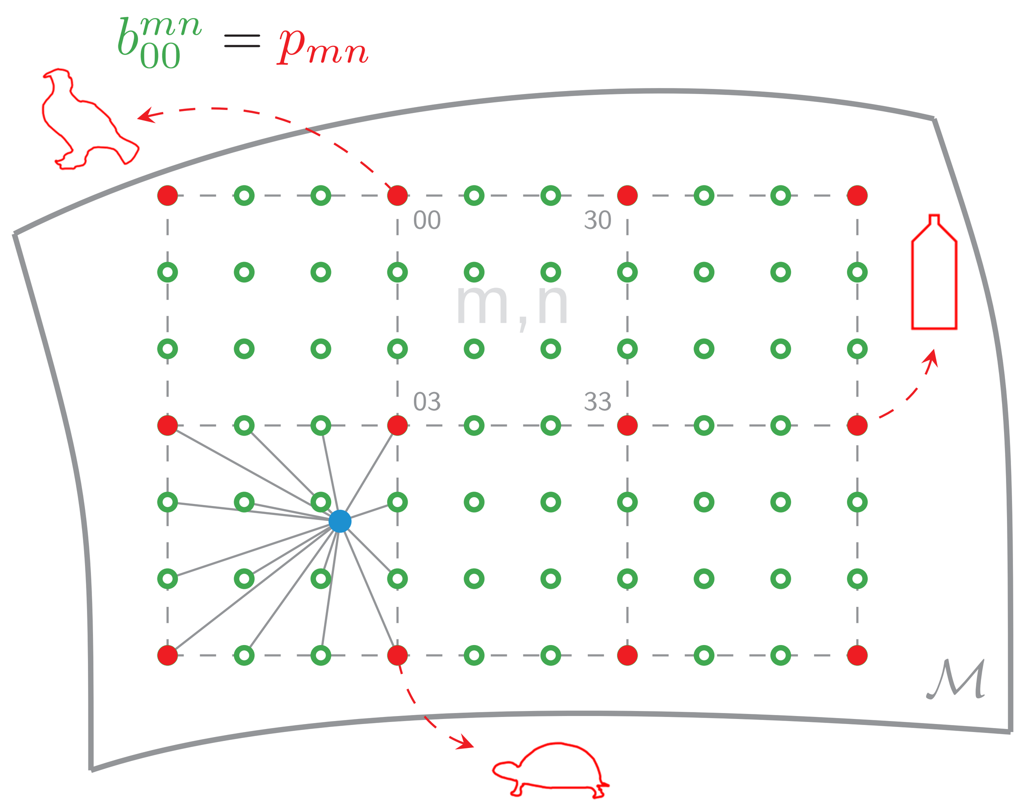
$$\beta(t_1,t_2,\mathbf{b}) = \mathrm{av}[\mathbf{b}, (w_{ij}(t_1,t_2))]$$

Convex/Karcher mean ← → Bernstein polynomials

**Smoothness constraint:** over $[0,M] \times [0,N]$, surface of minimial mean square acceleration. If $\mathcal{M} = \mathbb{R}^r$, $\mathfrak{B}$ is a natural cubic B-spline

$$\mathfrak{B} = \sum_{m,n} \alpha_{mn} B(t_1 - m) B(t_2 - n).$$

Advantage: Low space and time complexity!

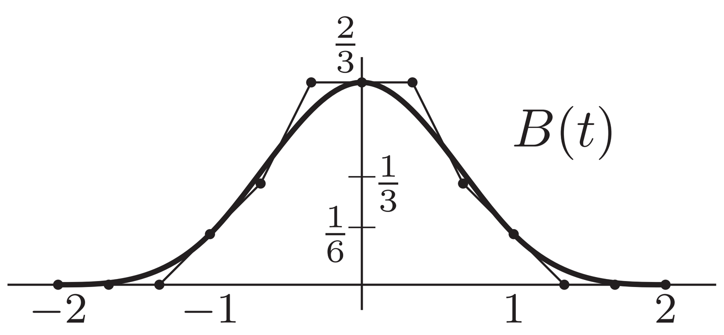



*e.g.*, $\mathcal{M}$ is the closed shape manifold

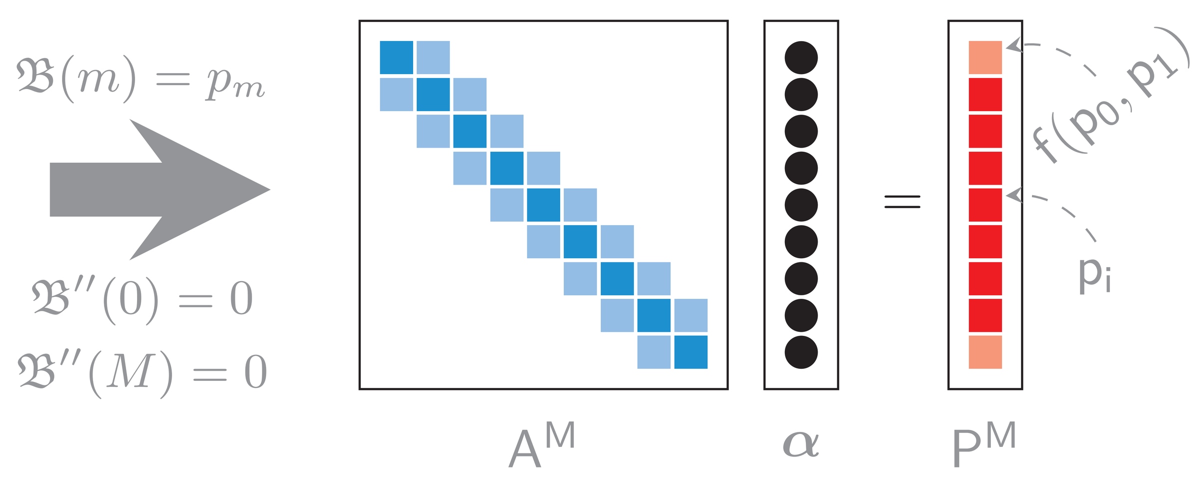
## Step 1: [Euclidean case] Find control points for curves (1D)

**Find $\mathbf{b}^m$ to interpolate $\{p_m\}$ with a piecewise Bézier curve $\mathfrak{B} = \beta(t - m, \mathbf{b}^m)$ with $m = \lfloor t \rfloor$**

$\mathfrak{B}$ is a natural cubic B-spline $\mathfrak{B} = \sum_{m=0}^{M-1} \alpha_m B(t-m)$ (smoothness constraint).

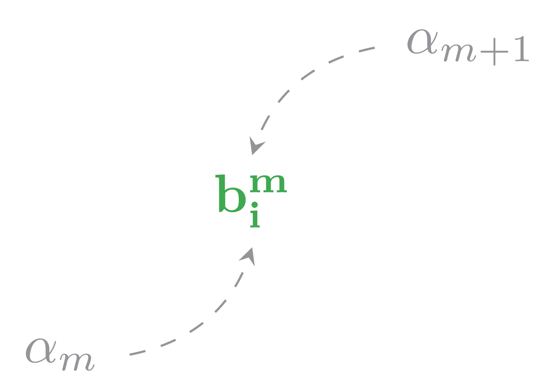



Under interpolation constraints, we find the B-spline coefficients $\alpha_m$.




convex $\alpha$ combination!

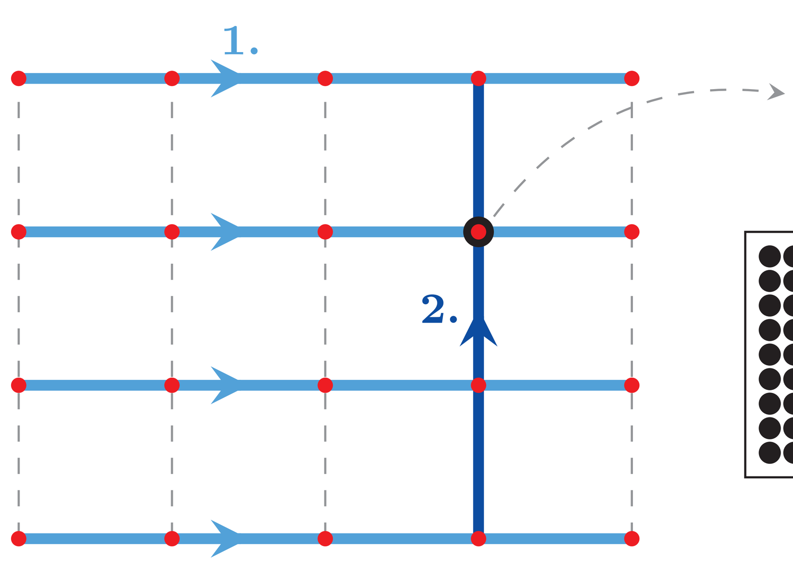
The control points $\mathbf{b}^m$ are convex combinations of $\alpha_m$.



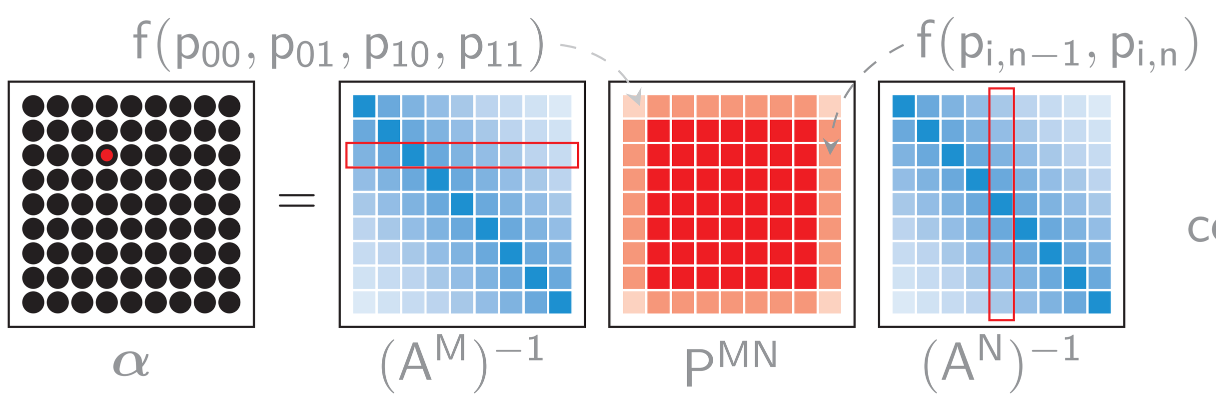

## Step 2: [Euclidean case] Use step 1 to find control points for surfaces (2D)

**Find $\mathbf{b}^{mn}$ to interpolate $\{p_{mn}\}$. Use step 1 twice to compute the coefficients $\alpha_{mn}$**

The surface $\mathfrak{B}$ is a tensorized version of the curve in step 1. We use step 1 in direction $t_1$ then $t_2$ to obtain the coefficients $\alpha_{mn}$.



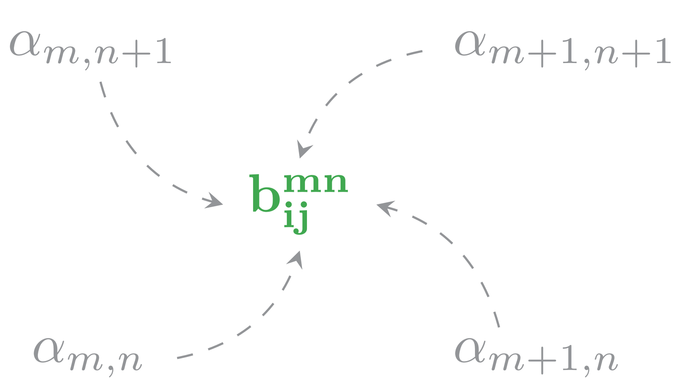

$$\alpha_{mn} = \sum_{i=1}^{M} \sum_{j=1}^{N} (A^M)^{-1}_{mi} (A^N)^{-1}_{nj} P_{ij}^{MN}$$




Accelerate by choosing only a few elements around the diagonal of $(A^\bullet)^{-1}$.

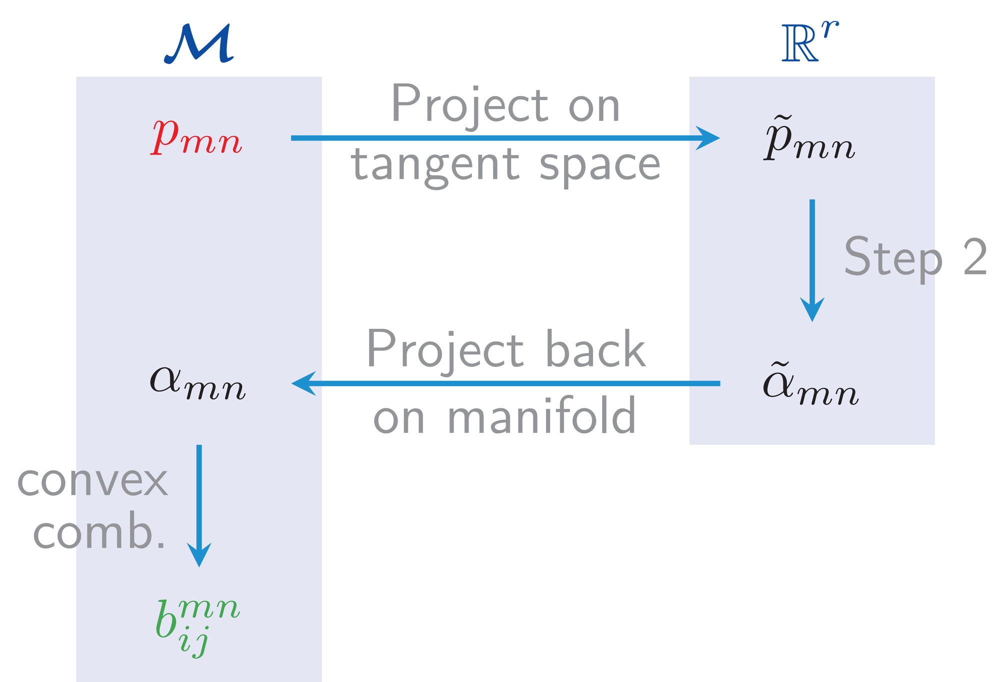
convex $\alpha$ combination!

The control points $\mathbf{b}^{mn}$ are convex combinations of $\alpha_{mn}$.



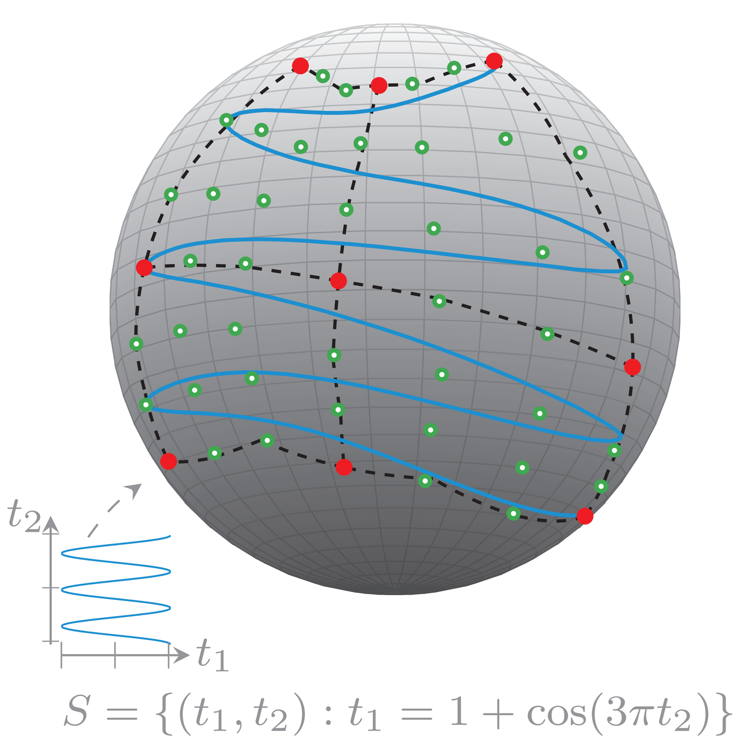

## Step 3: Generalize to manifolds

The problem is invariant to translations on $\mathbb{R}^r$.
1. Shift all data points $\{p_{mn}\}$ involved into the computation of $b_{ij}^{mn}$ to closest data point $p_{\mathrm{ref}}$.
2. Interpret $p_{mn} - p_{\mathrm{ref}}$ as a projection of $p_{mn} \in \mathcal{M}$ into the (Euclidean) tangent space of $p_{\mathrm{ref}}$.



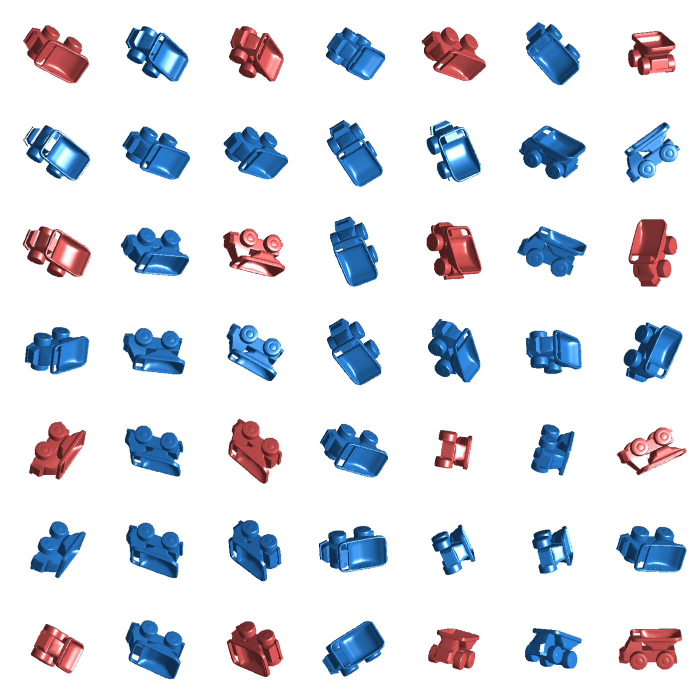

## Results

$\mathcal{M} = \mathbb{S}^2$




Smooth surface on the sphere interpolating the data points (red).

$\mathcal{M} = SO(3)$



Smooth surface on $\mathcal{M} = SO(3)$ interpolating rigid rotations (red).

## Summary

**Goal**
Find the control points $\{b_{ij}^{mn}\} \subset \mathcal{M}$ of the Bézier surface $\mathfrak{B}$ such that $\mathfrak{B}$ is the natural interpolating cubic B-spline when $\mathcal{M} = \mathbb{R}^r$. This ensures $\mathfrak{B}$ to minimize its mean square acceleration.

Project on tangent space

With data points $(p_{mn})$ compute $\hat{\alpha}_{mn}$ for curves in direction $t_1$ $\forall n$.

With $\hat{\alpha}_{mn}$ compute $\alpha_{mn}$ in direction $t_2$.

Project on manifold

**Compute** the control points $(b_{ij}^{mn}) \in \mathcal{M}$ as a convex combination of $\alpha_{mn}$.

**Reconstruct** each Bézier surface with weighted averaging

$$\beta = \mathrm{av}[\mathbf{b}^{mn}, (w_{ij})]$$

known on manifolds as the Karcher mean (generalization of the convex mean on $\mathbb{R}^r$).