Piecewise-Bézier C^1 interpolation on Riemannian manifolds with application to 2D shape morphing

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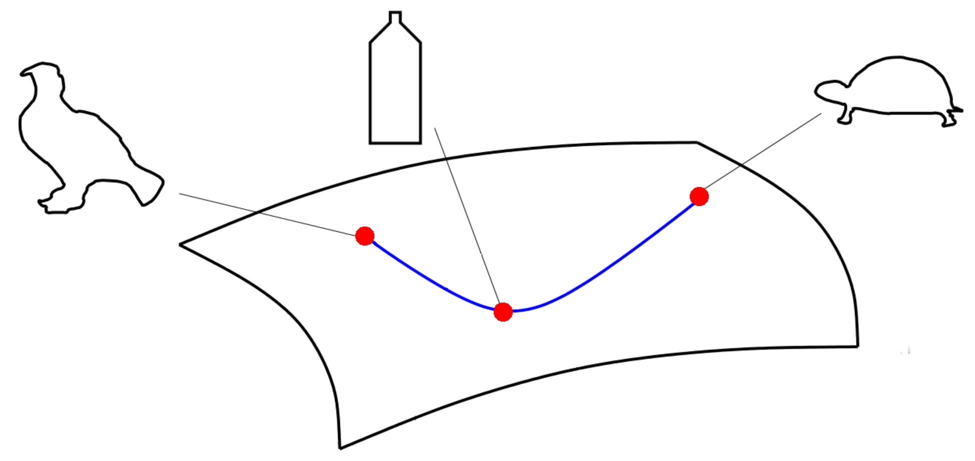
Context

What? A new framework to fit a smooth path to a finite set of given data points on a Riemannian manifold.
Why? Several applications in vision, such as reconstructing object shape evolution in time.
How? With a C^1-path of minimal square acceleration on the manifold, ensuring low space and time complexity.



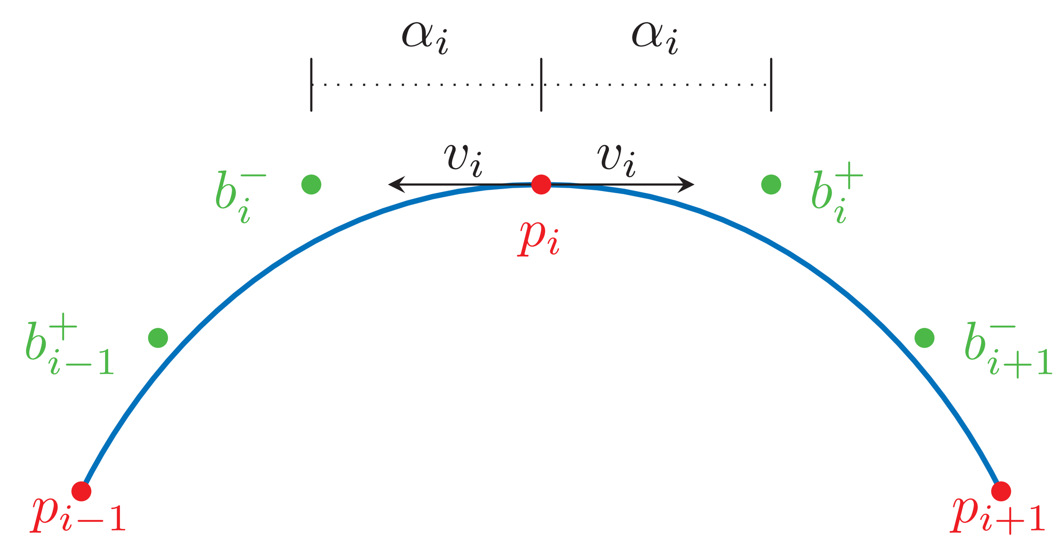
Conclusion

Formulation: An optimization problem solved by a tridiagonal linear system based on tools from differential geometry (fast);
Storage: Only $3n-1$ control points of the Bézier segments (light);
Solution: C^1-interpolation smooth path composed of Bézier functions (new).

Methodology on the Euclidean space

Input:

- $n+1$ data points $(p_0, \dots, p_n)$;
- $n-1$ velocity directions at internal data points $(v_1, \dots, v_{n-1})$;
- n Bézier functions driven by unknown intermediate control points: functions of degree 2 for the segment joining p_0 and p_1, as well as the segment joining p_{n-1} to p_n, and functions of degree 3 for the other segments.



Free variables: The norms α_i (scalar) of the velocity directions.

Constraint: The piecewise path is smooth at data points.

β_k: Bézier segment driven by p_{n-1}, b_{n-1} and p_n

$$\min_{\alpha_1,\dots,\alpha_{n-1}} \int_0^1 \|\ddot\beta_2(t; p_0, b_1^-, p_1)\|^2 \mathrm{d}t + \sum_{i=1}^{n-2} \int_0^1 \|\ddot\beta_3(t; p_{i-1}, b_{i-1}^+, b_i^-, p_i)\|^2 \mathrm{d}t + \int_0^1 \|\ddot\beta_2(t; p_{n-1}, b_{n-1}^+, p_n)\|^2 \mathrm{d}t,$$

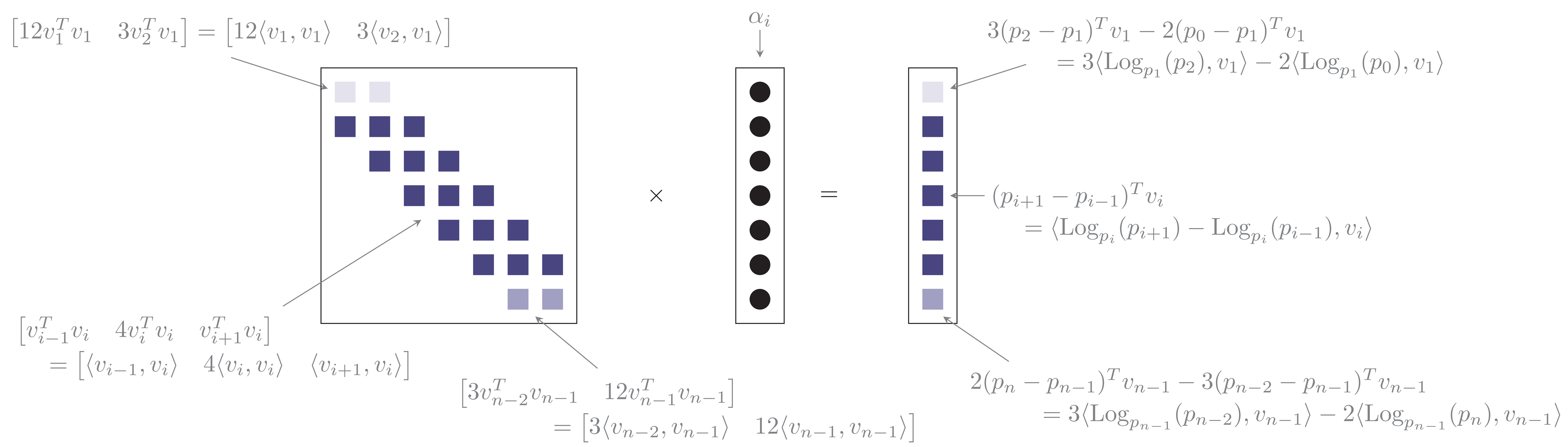
$b_1^- = p_1 - \alpha_1 v_1$ $\qquad$ $b_{i-1}^+ = p_{i-1} + \alpha_{i-1} v_{i-1}$

Output: The optimal intermediate control points yielding the piecewise Bézier path.

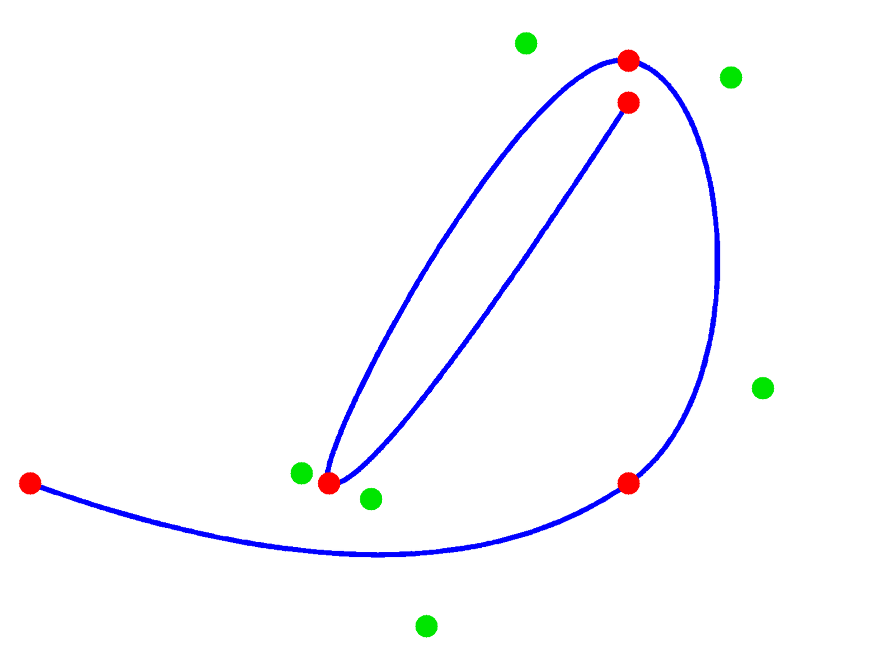
Reconstruction of the path: Application of the *De Casteljeau* algorithm generalized with geodesics to manifolds.

Mathematical formulation

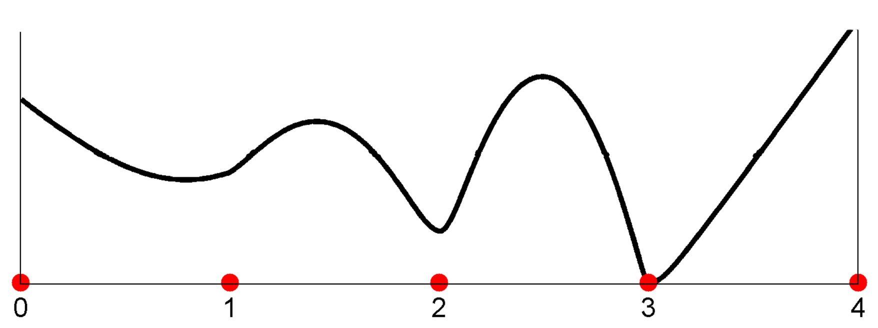
On Euclidean spaces, the minimization of the mean square acceleration of the piecewise path leads to a tridiagonal linear system with unknowns α_i. We generalize it to Riemannian manifolds with differential geometry tools as the scalar product ($a^T b = \langle a, b\rangle$) and the Logarithmic map ($b - a = \mathrm{Log}_a(b)$).



Result on the Euclidean plane

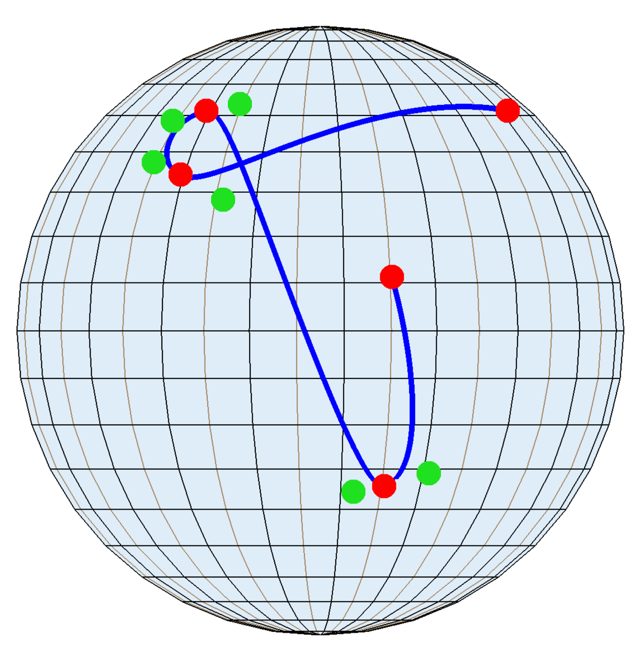


Norm of the velocity of the Bézier curve

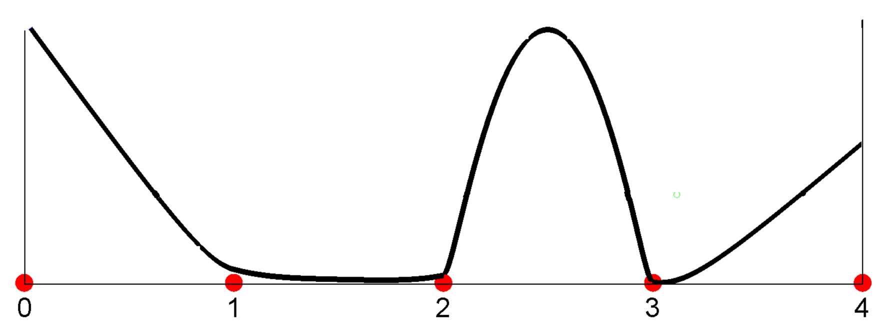


The optimization on a linear finite dimensional manifold produces a smooth path, as expected.

Result on the unit sphere

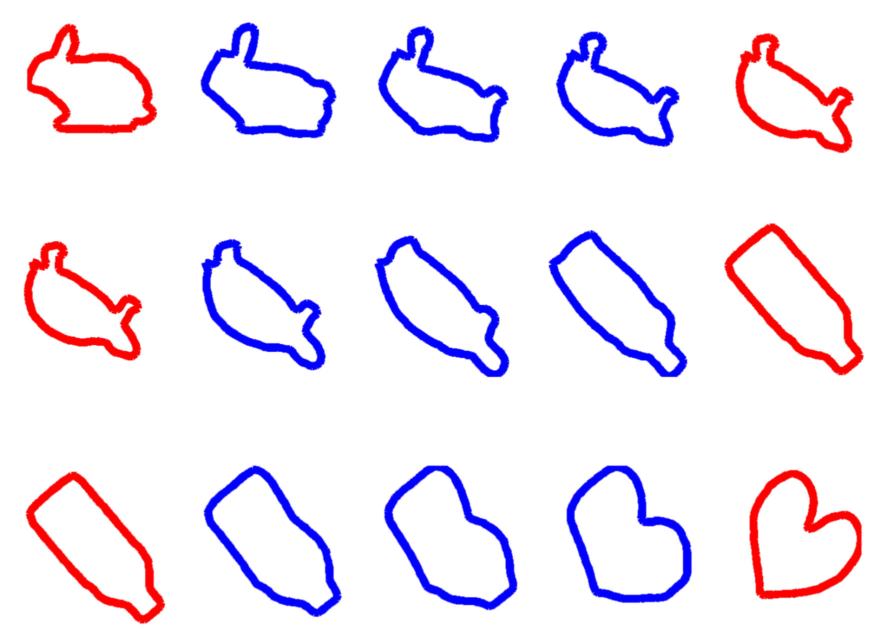


Norm of the velocity of the Bézier curve

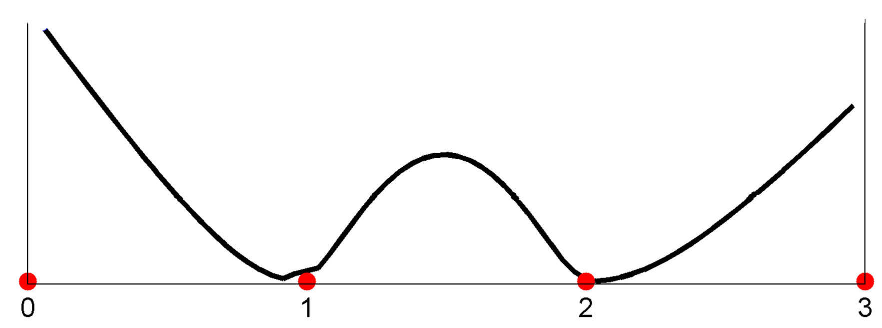


The generalization to a non-linear finite dimensional manifold leads to a smooth path.

Morphing of shapes



Norm of the velocity of the Bézier curve



The generalization to a more general manifold leads to visually satisfying results.