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Interpolation on the manifold of fixed-rank PSD matrices for Parametric Model Order Reduction: preliminary results

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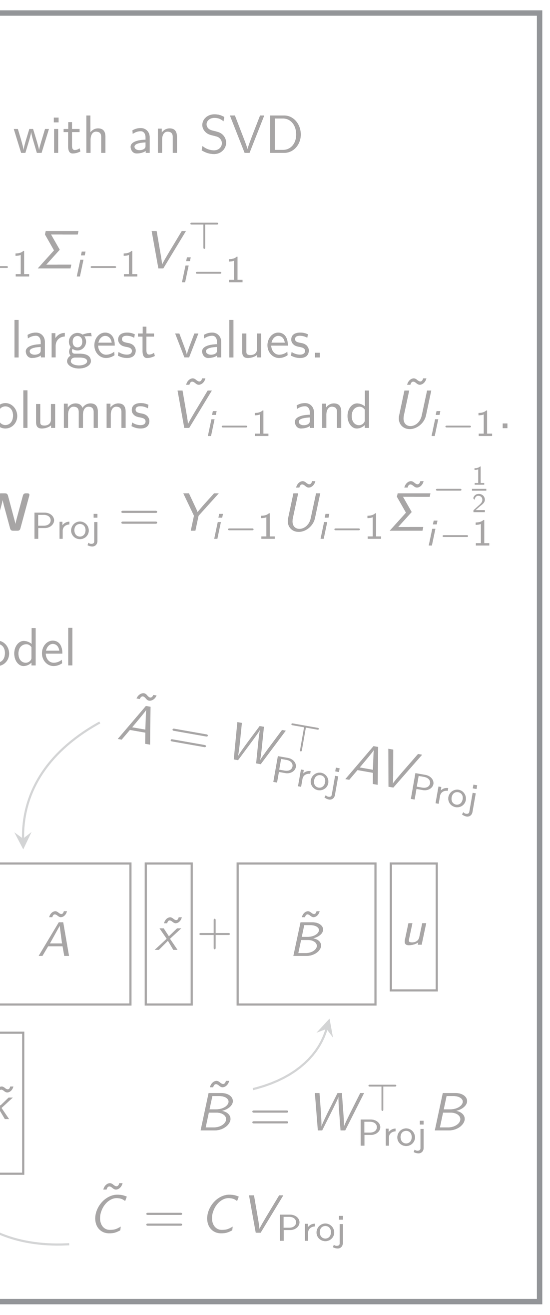
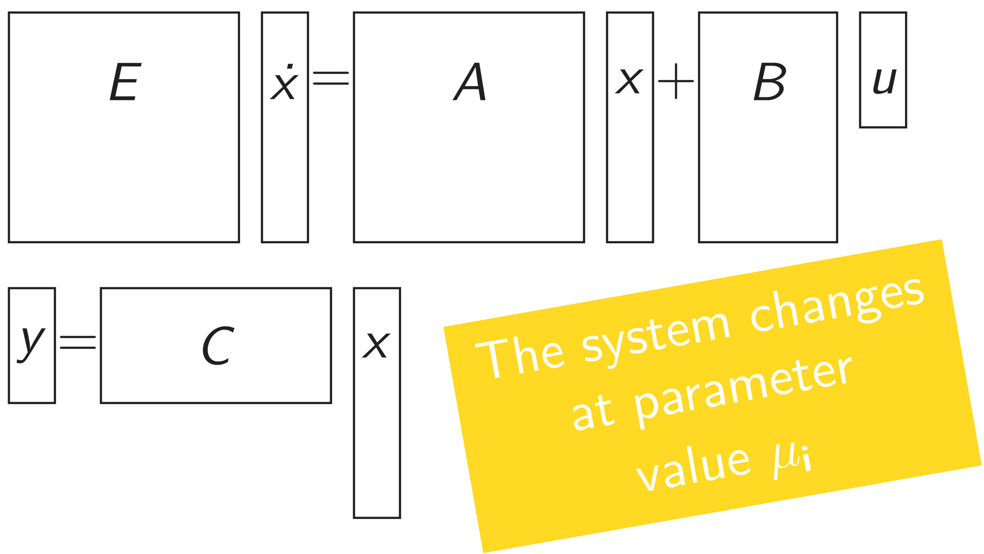


PMOR (balanced truncation)



1. Find the low rank solutions $P_i = X_i X_i^\top$, $Q_i = Y_i Y_i^\top$ of the Lyapunov equations:

$$\begin{cases} E P_i A^\top + A P_i E^\top = -BB^\top \\ E^\top Q_i A + A^\top Q_i E = -CC^\top \end{cases}$$

$X_i \in \mathbb{R}^{n\times k_{X_i}}$, $Y_i \in \mathbb{R}^{n\times k_{Y_i}}$ — truncate! → $P_i \in \mathcal{S}_+(p,n)$, $p = \min_i(k_{X_i})$; $Q_i \in \mathcal{S}_+(q,n)$, $q = \min_i(k_{Y_i})$

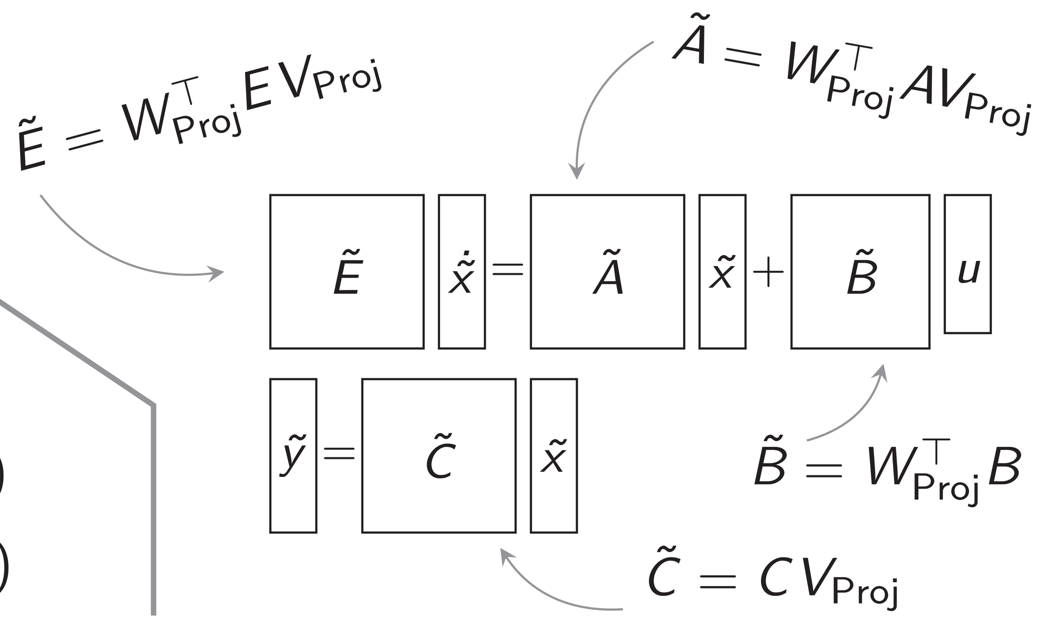
2. Find projectors V_{Proj} and W_{Proj} with an SVD

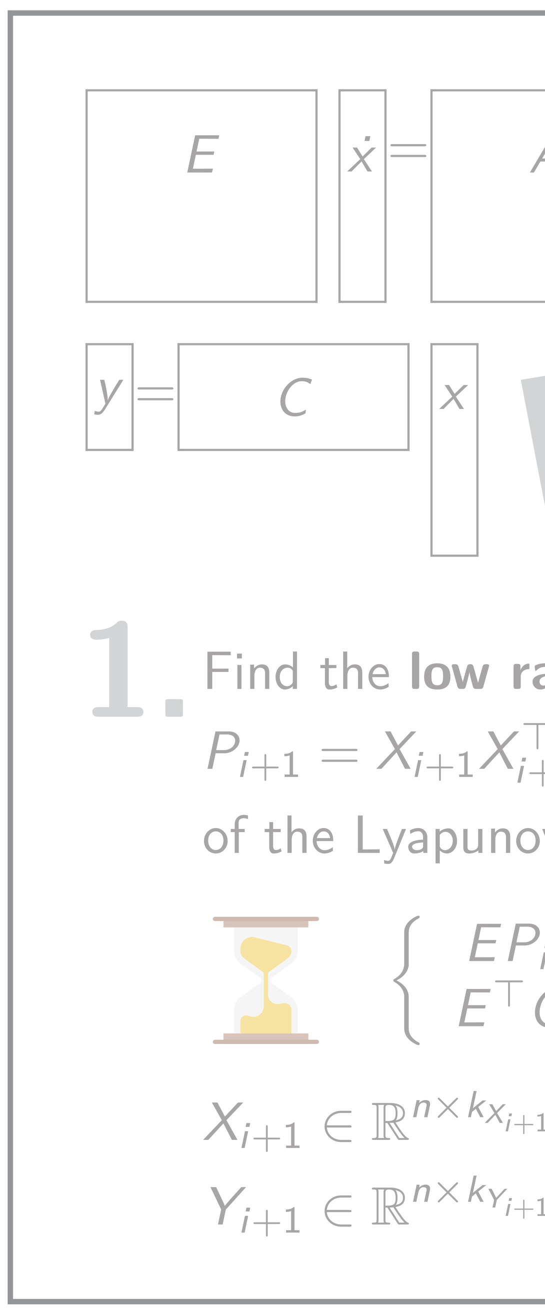
2.1. SVD step: $Y_i^\top E X_i = U_i \Sigma_i V_i^\top$

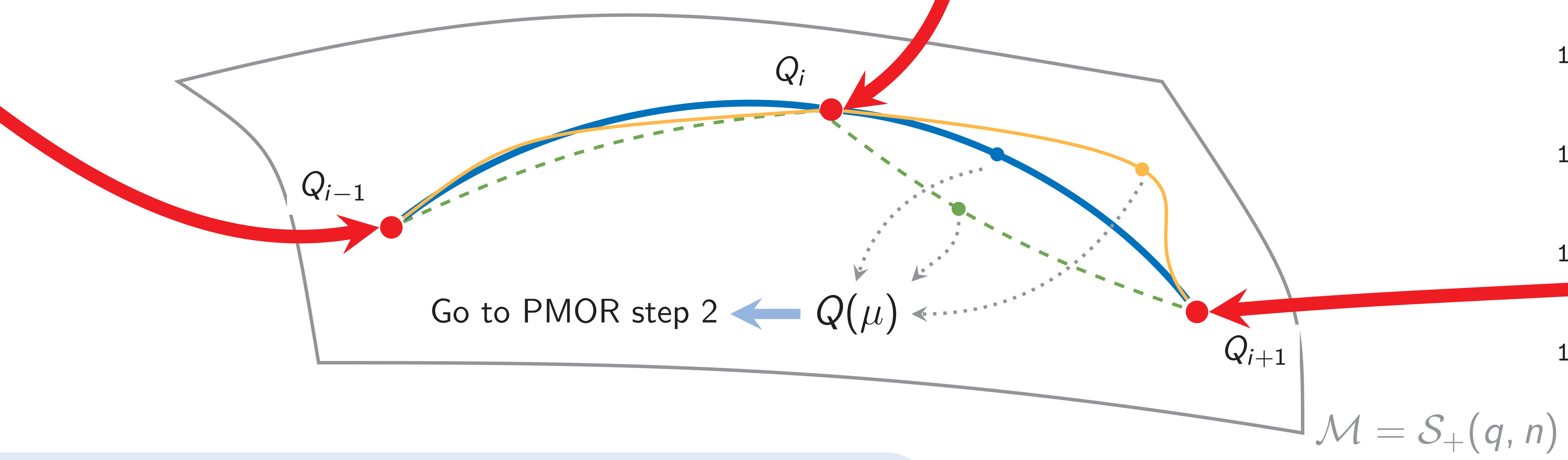
2.2. Σ_i truncated as $\tilde{\Sigma}_i$ to r largest values. V_i and U_i truncated to r first columns $\tilde{V}_i$ and $\tilde{U}_i$.

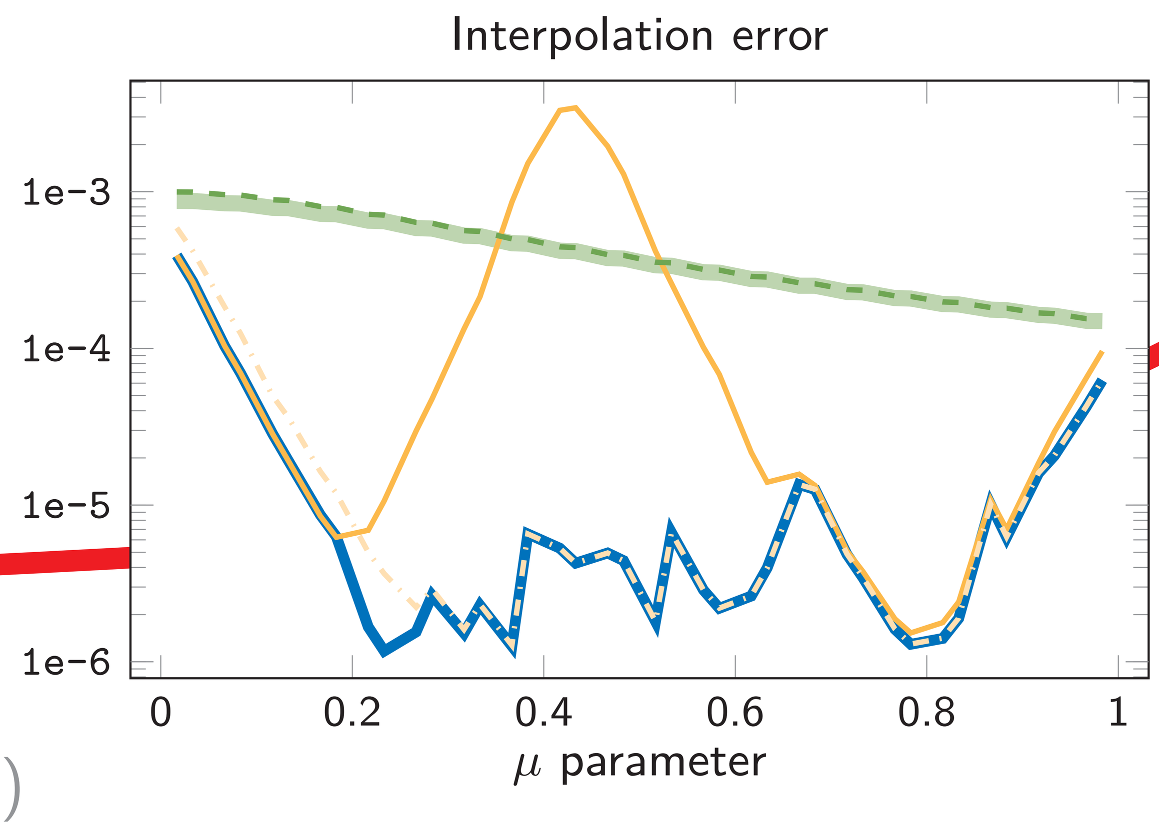
2.3. $V_{Proj} = X_i \tilde{V}_i \tilde{\Sigma}_i^{-\frac{1}{2}}$ and $W_{Proj} = Y_i \tilde{U}_i \tilde{\Sigma}_i^{-\frac{1}{2}}$

3. **The end-goal:** reduce the model



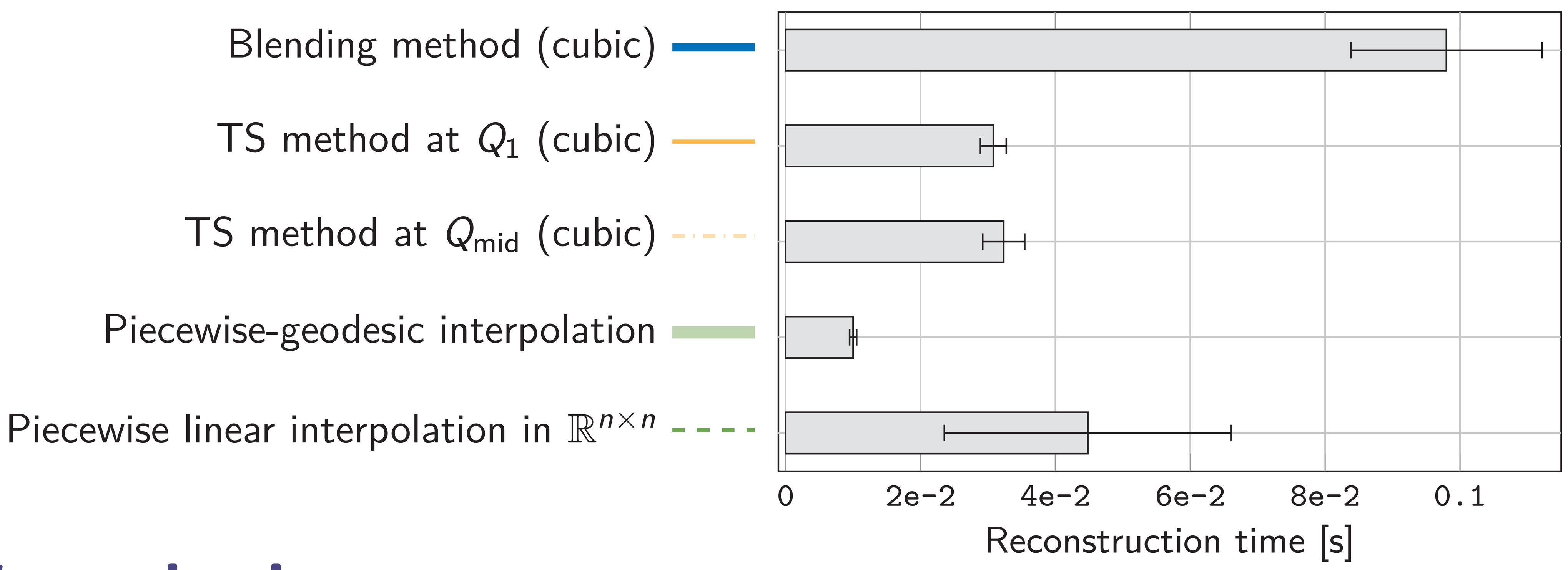






Goal: accelerate PMOR with interpolation.

- ✓ Evaluate matrices $\{P_0, \dots, P_i, \dots, P_m\} \in \mathcal{S}_+(p,n)$ and $\{Q_0, \dots, Q_i, \dots, Q_m\} \in \mathcal{S}_+(q,n)$ for a few values of the parameter $\mu_0, \dots, \mu_i, \dots, \mu_m \in \mathbb{R}$
- ✓ Recover matrices P and Q at a new value μ with interpolation on $\mathcal{S}_+(\cdot,n)$.

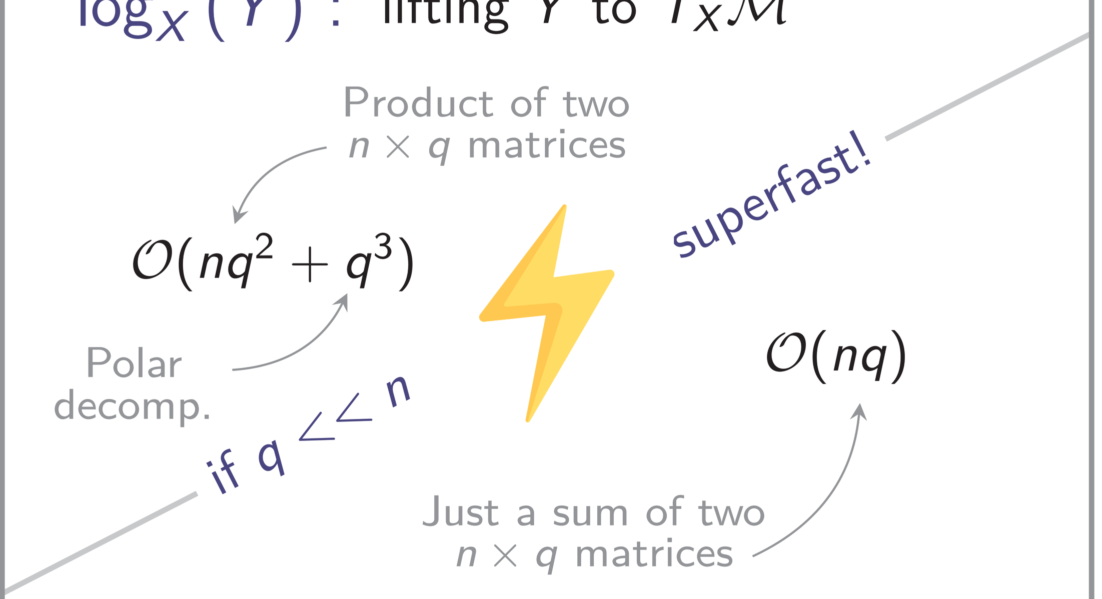


$\mathcal{M} = \mathcal{S}_+(q,n)$

$\mathcal{S}_+(q,n)$: space of positive-semidefinite matrices of size n and rank q

$X, Y \in \mathcal{M}$ — Manifold structure — $H \in T_X\mathcal{M}$
⇓
Equiped with **exp** and **log** maps
Near X, (Euclidean) tangent space $T_X\mathcal{M}$

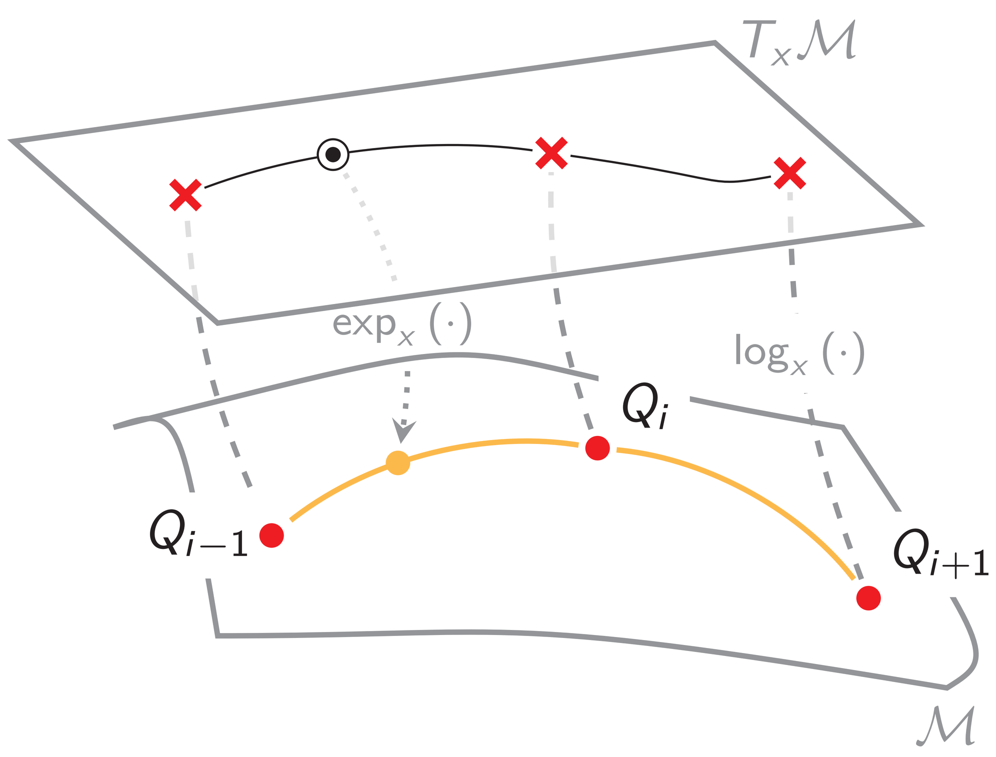
$\log_X(Y)$: lifting Y to $T_X\mathcal{M}$



mapping H to $\mathcal{M}$: $\exp_X(H)$

TS method

1. Lift all points Q_i to a reference **T**angent **S**pace $T_x\mathcal{M}$ at $x \in \mathcal{M}$ with the $\log_x(Q_i)$.
2. Compute a (Euclidean) solution $\tilde{Q}_x^{TS}(\mu)$ with your favorite tool.
3. Map the solution back to $\mathcal{M}$ as $Q_x^{TS}(\mu) = \exp_x(\tilde{Q}_x^{TS}(\mu))$



Blending method

Idea: blend solutions from two different (but close) tangent spaces

$$Q(\mu) = \mathrm{av}\left[(1-w,w),(Q_{Q_i}^{TS}(\mu)),(Q_{Q_{i+1}}^{TS}(\mu))\right]$$