Blended smoothing splines on Riemannian manifolds

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***Abstract* – We present a method to compute a fitting curve $\mathbf{B}$ to a set of data points $d_0, \ldots, d_m$ lying on a manifold $\mathcal{M}$. That curve is obtained by blending together Euclidean Bézier curves obtained on different tangent spaces. The method guarantees several properties among which $\mathbf{B}$ is $\mathcal{C}^1$ and is the natural cubic smoothing spline when $\mathcal{M}$ is the Euclidean space. We show examples on the sphere S^2 as a proof of concept.**[1]

1 Introduction

We address the problem of curve fitting on a Riemannian manifold $\mathcal{M}$. From a set of data points $d_0, \ldots, d_m \in \mathcal{M}$ associated with times $t_0, \ldots, t_m$ on a given time-interval $[0, n]$, we seek a $\mathcal{C}^1$ curve $\mathbf{B} : [0, n] \to \mathcal{M}$ that is "sufficiently straight", while approximating "sufficiently well" the data points at the given times.

Curve fitting on manifold appears in several applications where denoising or resampling time-dependent data is required. For instance, in Arnould *et al.* [2], the evolution of an organ is observed by interpolating several contours of a tumoral tissue on a shape manifold. Regression is also of interest in problems where 3D rigid rotations of objects are involved, as in motion planning of rigid bodies or in computer graphics [9]. In that case, the manifold would be the special orthogonal group SO(3).

A widely used strategy to address the fitting problem in general is to encapsulate the fitting and straightness constraints in a single optimization problem

$$\min_{\gamma \in \Gamma} E_\lambda(\gamma) := \int_{t_0}^{t_m} \left\| \frac{\mathrm{D}^2 \gamma(t)}{\mathrm{d}t^2} \right\|_{\gamma(t)}^2 \mathrm{d}t + \lambda \sum_{i=0}^{m} \mathrm{d}^2(\gamma(t_i), d_i), \tag{1}$$

where Γ is an admissible set of curves γ on $\mathcal{M}$, $\frac{\mathrm{D}^2}{\mathrm{d}t^2}$ is the (Levi-Civita) second covariant derivative, $\| \cdot \|_{\gamma(t)}$ is the Riemannian metric at $\gamma(t)$, and $\mathrm{d}(\cdot, \cdot)$ is the Riemannian distance. The parameter λ permits to strike the balance between the regularizer $\int_{t_0}^{t_m} \| \frac{\mathrm{D}^2\gamma(t)}{\mathrm{d}t^2} \|_{\gamma(t)}^2 \mathrm{d}t$ and the fitting term $\sum_{i=0}^{m} \mathrm{d}^2(\gamma(t_i), d_i)$.

This problem has been tackled in different ways in the past few years. We cite for instance Samir *et al.* [10] that approached the solution of problem (1) with a manifold-valued steepest-descent method on an infinite dimensional Sobolev space equipped with the Palais-metric. In Boumal *et al.* [3], the search space is reduced to the product manifold $\mathcal{M}^M$, as the curve $\mathbf{B}$ is discretized in M points, and the covariant derivative from (1) is approached with finite differences on manifolds. A technique for regression based on unwrapping and unrolling has been recently proposed by Kim *et al.* [7]. Finally, we mention Lin *et al.* [8], who proposed a polynomial regression technique based on projections on tangent spaces.

The limit case when $\lambda \to \infty$ concerns interpolation. We cite here several works that solve this problem by means of Bézier curves [2, 1]. In those works, the search space Γ is reduced

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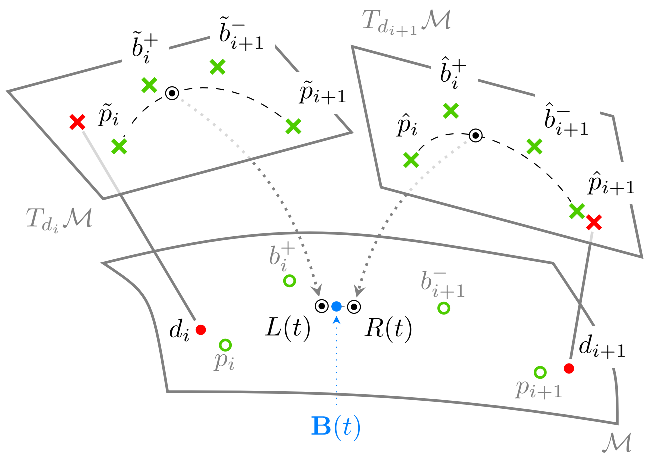


Figure 1: The curve $\mathbf{B}(t)$ is made of natural cubic splines computed on different tangent spaces. The cubic splines can be obtained equivalently as Bézier curves, using a technique close to [2]. They are then blended together with carefully chosen weights.

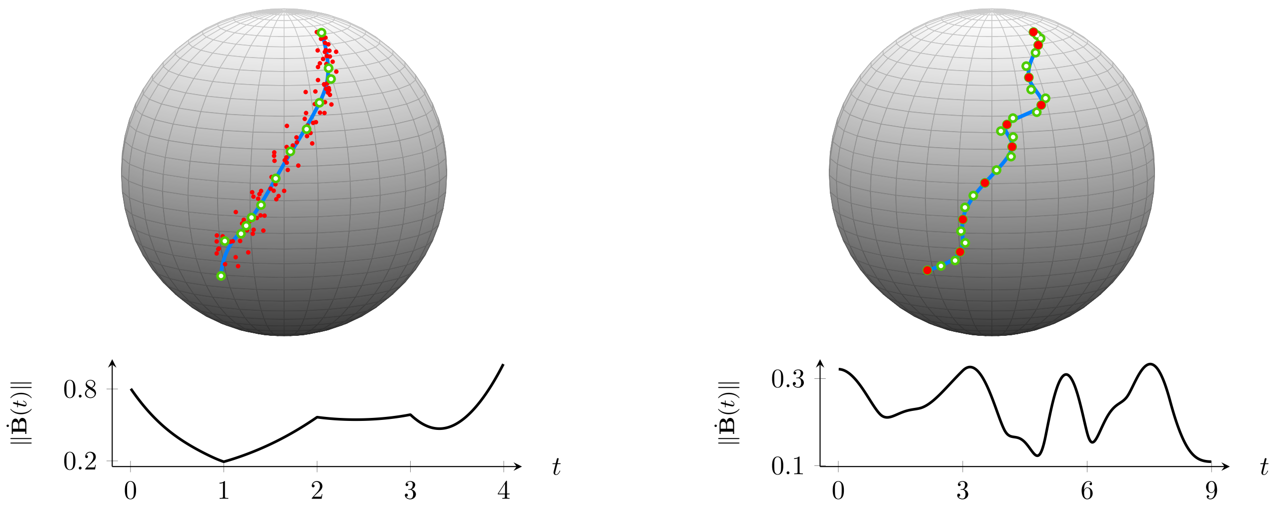
to composite cubic Bézier splines $\mathbf{B}$ and the optimality of (1) is guaranteed only when $\mathcal{M} = \mathbb{R}^r$. However, the main advantages of these methods are twofold: *(i)* the search space is drastically reduced to the so-called *control points* of $\mathbf{B}$ (see, e.g., [5] for an overview on Bézier curves); *(ii)* they are very simple to implement on any Riemannian manifold, as only two objects are required: the Riemannian exponential and the Riemannian logarithm, while most of the other techniques require a gradient or heavy computations of parallel transportation.

Our method aims to extend these works to fitting, and is extensively described in [6] for the case where $m = n$. We build several polynomial pieces by solving the problem (1) on carefully chosen tangent spaces, and then blend together these curves in such a way that $\mathbf{B}$ *(i)* is differentiable, *(ii)* is the natural cubic smoothing spline when $\mathcal{M}$ is a Euclidean space, *(iii)* interpolates the data points if $m = n$ when $\lambda \to \infty$. Furthermore, we assess that the method is easy-to-use, as *(iv)* it only requires the knowledge of the Riemannian exponential and the Riemannian logarithm on $\mathcal{M}$; *(v)* the curve can be stored with only $\mathcal{O}(n)$ tangent vectors; and, finally, *(vi)* given this representation, computing $\gamma(t)$ at $t \in [0, n]$ only requires $\mathcal{O}(1)$ exponential and logarithm evaluations.

We present here the above-mentioned method and give results for fitting on the sphere S^2. We refer to [6] for more details and for the proof of the six properties.

2 Method

Framework. Consider a Riemannian manifold $\mathcal{M}$ and a set of $m+1$ data points $d_0, \ldots, d_m \in \mathcal{M}$ associated with parameters $t_0, \ldots, t_m$ over an interval $[0, n]$. Our method relies on computations on tangent spaces. For this, we define the points $d(i)$, $i = 0, \ldots, n$, where $d(i) = d_{k_i}$ is the data point whose associated parameter t_{k_i} is the closest to $t = i$. We denote $T_{d(i)}\mathcal{M}$ its associated tangent space. Consider finally the search space Γ from (1) reduced to the space of $\mathcal{C}^1$ composite



(a) Smoothing curve $\mathbf{B} : [0, 4] \to \mathcal{M}$ fitting 100 data points.

(b) Fitting curve $\mathbf{B} : [0, 9] \to \mathcal{M}$, with $\lambda = 10^8$.

Figure 2: The data points (red dots) are fitted by a $\mathcal{C}^1$ composite blended spline $\mathbf{B}(t)$ (blue). The blended spline is here represented as a Bézier curve conducted by its control points (green circles).

curves

$$\mathbf{B} : [0, n] \to \mathcal{M} : f_i(t - i),\ i = \lfloor t \rfloor,$$

where the functions $f_i : [i, i+1] \to \mathcal{M}$ are called *blended functions*. They are given by

$$f_i(t - i) = \mathrm{av}[(L_i(t), R_i(t)), (1 - w(t), w(t))],$$

for $i = 0, \dots, n$ and where $\mathrm{av}[(x, y), (1 - a, a)]$ is a Riemaniann weighted mean. The fitting technique we present here consists in computing the functions $L_i(t)$, $R_i(t)$ and choosing the weight function $w(t)$ such that the six above-mentioned properties are met.

Optimal curves. The functions $L_i(t)$ and $R_i(t)$ are obtained as follows. We note $\tilde{x} = \mathrm{Log}_{d(i)}(x)$ and $\hat{x} = \mathrm{Log}_{d(i+1)}(x)$, the representation of the point $x \in \mathcal{M}$ in the tangent spaces at $d(i)$ and $d(i+1)$ respectively. We define $L_i(t) = \mathrm{Exp}_{d(i)}\left(\tilde{\mathbf{B}}(t)\right)$ and $R_i(t) = \mathrm{Exp}_{d(i+1)}\left(\hat{\mathbf{B}}(t)\right)$, where $\tilde{\mathbf{B}}(t)$ is the natural cubic spline fitting the data points $\tilde{d}_0, \dots, \tilde{d}_m$ on $T_{d(i)}\mathcal{M}$, and accordingly for $\hat{\mathbf{B}}(t)$. Note that $\tilde{\mathbf{B}}(t)$ (resp. $\hat{\mathbf{B}}(t)$) are therefore solutions of (1) on the corresponding tangent space.

Riemannian averaging. Finally, the choice of the weight function $w(t)$ is of high importance in order to meet the differentiability property. The weight function must thus be chosen such that $L_i(0) = f_i(0)$, $R_i(1) = f_i(1)$, $\dot{L}_i(0) = \dot{f}_i(0)$ and $\dot{R}_i(1) = \dot{f}_i(1)$. This is obtained for $w(1) = 1$, and $w(0) = w'(0) = w'(1) = 0$. Among all the possible weight functions, we choose $w(t) = 3t^2 - 2t^3$.

The blending method is represented in Figure 1.

3 Results

We show two examples on S^2. Figure 2a presents a smoothing curve fitting 100 noisy points at times $t_i \in [0, 4]$ with $\lambda = 100$. Figure 2b shows the fitting curve obtained for 10 data points at times $t_i = i$, $i = 0, \dots, 9$, for $\lambda = 10^8$. We observe in both cases that the curve is $\mathcal{C}^1$ (property *(ii)*) and that the data points are interpolated (property *(iii)*) when $\lambda \to \infty$. Property *(i)* is obtained by construction. Properties *(iv-vi)* are shown and proved in [6]. Additionnal examples on the special orthogonal group $\mathrm{SO}(3)$ or on the manifold of positive semidefinite matrices of size p and rank q, $\mathcal{S}_+(p, q)$, are also provided in [6].

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