Data fitting on manifolds with blended cubic splines

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We address the problem of curve fitting on a Riemannian manifold $\mathcal{M}$: given $n+1$ data points $d_0, \dots, d_n \in \mathcal{M}$, associated with real (time-)parameters $t_0, \dots, t_n$, we seek a curve $\gamma : [0, n] \to \mathcal{M}$ being, on the one hand, "sufficiently close" to the data points, while, on the other hand, being "sufficiently straight". A strategy to do so is to encapsulate the two above mentioned goals in an optimization problem

$$\min_{\gamma \in \Gamma} E_\lambda(\gamma) := \int_{t_0}^{t_n} \left\| \frac{\mathrm{D}^2\gamma(t)}{\mathrm{d}t^2} \right\|^2_{\gamma(t)} \mathrm{d}t + \lambda \sum_{i=0}^{n} \mathrm{d}^2(\gamma(t_i), d_i), \tag{1}$$

where Γ is an admissible set of curves γ on $\mathcal{M}$, $\frac{\mathrm{D}^2}{\mathrm{d}t^2}$ is the (Levi-Civita) second covariant derivative, $\| \cdot \|_{\gamma(t)}$ is the Riemannian metric at $\gamma(t)$, and $\mathrm{d}(\cdot, \cdot)$ is the Riemannian distance. The problem also has a parameter λ that strikes the balance between the two goals of the problem, i.e., the regularizer $\int_{t_0}^{t_n} \| \frac{\mathrm{D}^2\gamma(t)}{\mathrm{d}t^2} \|^2_{\gamma(t)} \mathrm{d}t$ and the fitting term $\sum_{i=0}^{n} \mathrm{d}^2(\gamma(t_i), d_i)$.

We present here a method that extends the work of (Arnould et al., 2015). In a nutshell, we reduce the search space of (1) to the space of C^1 composite curves

$$\mathbf{B} : [0, n] \to \mathcal{M} : f_i(t - i), \; i = \lfloor t \rfloor,$$

made of so-called blended functions f_i. These blended functions are given by

$$f_i(t) = \mathrm{av}[(L_i(t), R_i(t)), (1 - w(t), w(t))],$$

where $\mathrm{av}[(x, y), (1 - a, a)]$ is a weighted mean, $w(t) = 3t^2 - 2t^3$, and where $R_i(t)$ and $L_i(t)$ are cubic Bézier curves (Farin, 2002) computed respectively on $T_{d_i}\mathcal{M}$ and $T_{d_{i+1}}\mathcal{M}$, $i = 0, \dots, n-1$, with the control points optimized with a technique similar to (Arnould et al., 2015). The blending method is represented in Figure 1.

The method guarantees the five following properties: (i) the curve is C^1 on $[t_0, t_n]$; (ii) the curve interpolates the data points $d_0, \dots, d_n$ when $\lambda \to \infty$; (iii) the curve is the natural cubic spline minimizing (1) over a Sobolev space $H^2(t_0, t_n)$ when $\mathcal{M}$ is a Euclidean space; (iv) the method is designed for ease to use: it only requires the knowledge of the Riemannian exponential and the Riemannian logarithm on $\mathcal{M}$; (v) the curve can be stored with only $\mathcal{O}(n)$ tangent vectors; and, finally, (vi) given this representation, computing $\gamma(t)$ at $t \in [t_0, t_n]$ only requires $\mathcal{O}(1)$ exp and log evaluations.

Further details will be available in (Gousenbourger et al., 2018).

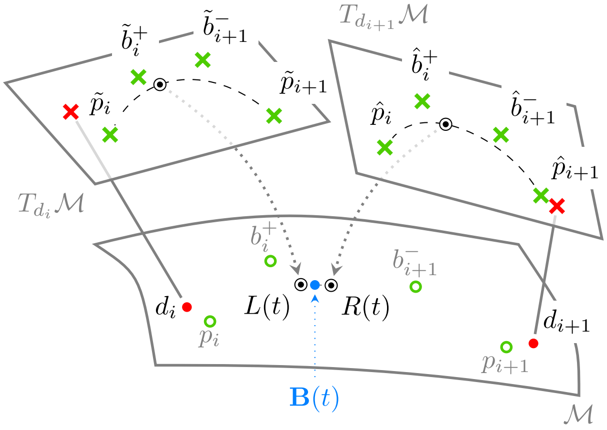

Figure 1. The composite curve $\mathbf{B}(t)$ is made of cubic Euclidean Bézier curves computed on different tangent spaces, and then blended together with carefully chosen weights.

References

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