Wind field estimation via C^1 Bézier smoothing on manifolds

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1 Introduction

The problem of fitting a regression curve to manifold-valued data is formulated. This problem has applications in any field where denoising or resampling is required. We consider the general case in which the data points $(d_0, \dots, d_n)$ are evaluated on a manifold $\mathcal{M}$ and are associated to measurement parameters $t_0 \leq \cdots \leq t_n$. We also consider that the curve is a smooth (*i.e.*, $\mathcal{C}^1$) composite Bézier curve $\mathfrak{B} : \mathbb{R} \to \mathcal{M}$ to reduce the search space. This abstract summarizes the method proposed in [1].

A very close problem is *interpolation*, already applied to parametric model reduction problems [2] where the data points belong to the Grassmann manifold. We illustrate the benefits of our *fitting* method by estimating wind fields characterized by convariance matrices, *i.e.*, data points belonging to $\mathcal{S}_+(r, p)$, the manifold of $p \times p$ positive semidefinite (PSD) matrices of rank r. These data points (d_i) are estimated based on solutions of computationally expensive CFD simulations. They are associated to some physical parameters, *e.g.*, prevaling wind orientations or magnitude (t_i). The idea is to run the simulations for a few values of the parameter and then estimate the covariance matrices via the fitted curve for other prevailing winds. We show that our method succeeds in approaching the ground truth when the data is corrupted by noise (Figure 1).

2 Bézier smoothing on manifolds

On the Euclidean space $\mathbb{R}^r$, the ith *Bézier curve of degree* $K \in \mathbb{N}$ in the fitting composite curve $\mathfrak{B}$ is a function parametrized by *control points* $b_0^i, \dots, b_K^i \in \mathbb{R}^r$ of the form

$$\beta_K^i(\cdot; b_0^i, \dots, b_K^i) : [0,1] \to \mathbb{R}^r, t \mapsto \sum_{j=0}^{K} b_j^i B_{jK}(t), \quad (1)$$

where $B_{jK}(t) = \binom{K}{j} t^j (1-t)^{K-j}$ are Bernstein polynomials (also called binomial functions) [3]. Bézier curves can be generalized to a Riemannian manifold $\mathcal{M}$ via the De Casteljau algorithm, which only requires the Riemannian exponential and logarithm to replace the straight line.

We construct the piecewise-Bézier curve in the same manner as in [4, (5)] but we relax the interpolation constraint by adding a data fidelity term. The problem becomes $\min_{b_K^i} f(\mathfrak{B}) + \lambda \sum_{i=0}^{n} \mathrm{d}^2(p_i, d_i)$ where the parameter λ adjusts the balance between data fidelity and the "smoothness" of $\mathfrak{B}$, $p_i = b_0^i = b_K^{i-1}$ for applicable i and $f(\mathfrak{B})$ is the mean square acceleration of $\mathfrak{B}$. In $\mathbb{R}^r$, in view of the translation invariance of the problem, the optimal control points can be expressed as affine combinations of the data points. We generalize this result to manifolds with the technique developped in [4].

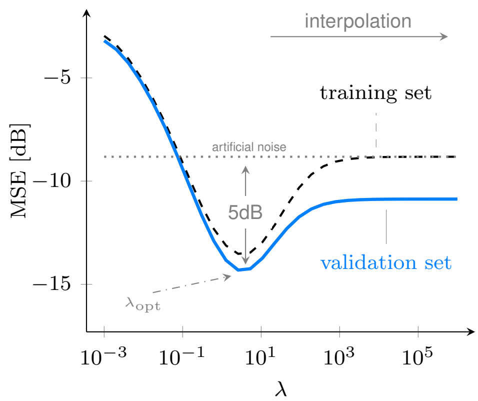

Figure 1: Mean squared error (MSE) obtained on training and validation sets, with artificial noise of 8 dB added to the data. Our method shows denoising capacities (here we observe 5 dB of MSE reduction compared to the noise level).

References

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