Differentiable 2D Bézier interpolation on manifolds

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1 Introduction

This abstract introduces bivariate interpolation of manifold-valued data points p_{ij}. The considered manifold is denoted by $\mathcal{M}$. The data points are associated with nodes $(i,j) \in \mathbb{Z}^2$ of a Cartesian grid in $\mathbb{R}^2$. In the present work, we seek a bivariate piecewise-cubic $\mathcal{C}^1$ Bézier function $\mathfrak{B} : \mathbb{R}^2 \to \mathcal{M}$ such that $\mathfrak{B}(i,j) = p_{ij}$.

Several applications motivate this problem, such as projection-based model order reduction of a dynamical system depending on a few parameters (where $\mathcal{M}$ is a Grassmann manifold) [3].

A detailed report about the work proposed here is available in [1].

2 Differentiable bivariate Bézier splines

In a Euclidean space $\mathbb{R}^r$, cubic Bézier surfaces are functions of the form

$$\begin{aligned} \beta_3(\cdot,\cdot;(b_{ij})_{i,j=0,\dots,3}) &: [0,1]^2 \to \mathbb{R}^r, \\ (t_1,t_2) &\mapsto \textstyle\sum_{i,j=0}^{K} b_{ij} B_{i3}(t_1) B_{j3}(t_2), \end{aligned}$$

where $B_{j3}(t) = \binom{3}{j} t^j (1-t)^{3-j}$ are Bernstein polynomials. The Bézier surfaces are parameterized by *control points* $(b_{ij})_{i,j=0,\dots,3} \subset \mathbb{R}^r$: they define how the surface spatially behaves and are interpolated when their indices i,j are both in $\{0,3\}$.

A bivariate Bézier spline corresponds to several Bézier surfaces β_3^{mn} patched together respectively in the x and y direction. They are continuously patched if, at the shared border of the two surfaces, their control points are exactly the same. Furthermore, on the Euclidean space, a simple linear constraint on the control points suffices to achieve the differentiability of the spline.

It is quite simple to extend the definition of Bézier surfaces to manifolds (we propose different techniques leading to slightly different results), as well as the continuity condition. However, differentiability no longer holds by simply generalizing the simple Euclidean constraint. We hence introduce a modified definition of the Bézier surfaces such that we obtain $\mathcal{C}^1$ splines on any Riemannian manifold.

3 Optimal splines

We optimize the control points under the $\mathcal{C}^0$ and $\mathcal{C}^1$ constraints such that the mean square acceleration of the surface is minimized when the manifold is the Euclidean space $\mathbb{R}^r$. In other words, we minimize the objective function

$$f[b_{ij}^{mn}] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{F}[\beta_3^{mn}]$$

where $\hat{F}[\beta_3^{mn}]$ is the energy of the Bézier function on the patch (m,n). In $\mathbb{R}^r$, in view of the translation invariance of the problem, the optimal control points can be expressed as affine combinations of the data points. We generalize this result to the manifold setting as a linear expression in a given tangent space, using a technique close to the one developed in [2].

We illustrate our method with surfaces computed on several manifolds such as the sphere or the special orthogonal group $SO(3)$ in order to interpolate rigid body positions.

References

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